Model-Based Seasonal Adjustment Diagnostics

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Abstract

Several model-based seasonal adjustment diagnostics are currently being incorporated into X-13-ARIMA-SEATS, a new hybrid seasonal adjustment program under development at the U.S. Census Bureau. These diagnostics have been implemented in the Ox programming language, and this paper discusses various aspects of the implementation. The diagnostics are normalized quadratic forms in the data, and roughly correspond to intra- and inter-component variation.

1 Introduction

A natural way to assess the quality of signal extraction is to check the spectrum of the estimated noise for residual signal. However, it is difficult to quantify any given form of assessment without making further assumptions about the process. One approach is to stipulate a model for the data, and gauge the degree to which the spectrum for the estimated noise deviates from what the model predicts. In the model-based signal extraction scenario such an approach – formulated in the time domain – is presented in Maravall (2003), which first proposed a single statistic that would quantify the goodness of signal extraction. Maravall's diagnostic was earlier incorporated into SEATS, the widely-used model-based seasonal adjustment program of Gómez and Maravall (1994), but wasn't documented until Maravall (2003). The basic concept of the diagnostic was modified and studied in Findley, McElroy, and Wills (2004), adapting the idea for finite sample signal extraction. The concept of the intra-component diagnostic is to measure the variation of an estimated component, assessed through a variance estimate of the appropriately "differenced" signal extraction estimate, and compare this quantity to what we would expect if our model were true. The inter-component diagnostic assesses variability between estimated components in the same manner. Extreme values of the statistic indicate model inadequacy – and hence filter inadequacy – with respect to the component model for the desired signal. Findley et al. (2004) discuss the basic theory of the new signal extraction diagnostics, and describe the results of simulation and empirical studies, while McElroy (2005b) expands and generalizes these ideas, providing asymptotic results. The paper at hand relates the implementation of these methods in the Ox (Doornik, 1998) program *SigDiagObj.ox*, utilizing the SsfPack function suite (Koopman, Shepherd, and Doornik, 1999).

Findley, et al. (2004) principally dealt with simple seasonal models, and focused on the irregular component. The current implementation considers a greater variety of components, and a broader class of models. In particular, we consider data decompositions of the form

$$Y_t = S_t + T_t + I_t, \tag{1}$$

where the data is Y_t, the seasonal is S_t, the trend-cycle is T_t, and the irregular is I_t. The program *SigDiagObj.ox* conducts four basic operations: estimation, decomposition, signal extraction, and diagnostic calculation. This paper discusses each of these steps in some detail, illustrated through the series of U.S. Retail Sales of Shoe Stores data from the monthly Retail Trade Survey of the Census Bureau, from 1984 to 1998, which will be referred to as the "shoe" series.

2 Estimation

In order to do model-based signal extraction, it is necessary to specify models for each of the components. We will follow the canonical decomposition approach of Hillmer and Tiao (1982), and so we start with a Seasonal ARIMA (SARIMA) model for monthly data Y_t, after suitable Box-Cox transformations:

$$\phi(B)\Phi(B^{12})(1-B)^d(1-B^{12})^D Y_t = \theta(B)\Theta(B^{12})\epsilon_t^Y \qquad t = 1, 2, \cdots, n \tag{2}$$

where $\phi(z)$, $\Phi(z)$, $\theta(z)$, and $\Theta(z)$ are polynomials with roots outside the unit circle of the complex plane, ensuring an invertible representation for the differenced data. Here ϵ_t^Y is a white noise innovation sequence. We allow for d nonseasonal differences and D seasonal differences. Currently the program works with $d, D \leq 1$ but not both zero; this is not a real restriction in practice, since these differencing orders allow for a fairly wide range of nonstationary processes. The assumed component models are given as follows:

$$\begin{aligned} \Phi(B^{12})U(B)^D S_t &= \theta^S(B)\epsilon_t^S \\ \phi(B)(1-B)^{d+D} T_t &= \theta^T(B)\epsilon_t^T \\ I_t &= \epsilon_t^I, \end{aligned} \tag{3}$$

where the various ϵ_t sequences are independent white noise. As above, the MA polynomials $\theta^S(z)$ and $\theta^T(z)$ are chosen so as to guarantee an invertible representation. The nonseasonal AR polynomial is associated with the trend-cycle component T_t, and the seasonal AR goes with the seasonal component. The polynomial $U(z) = 1 + z + z^2 + \cdots + z^{11}$ achieves seasonal differencing, and satisfies $1 - z^{12} = U(z)(1-z)$. The irregular component is assumed to be white noise. The canonical decomposition method begins by estimating all the parameters (including the innovation variance $\sigma^2_{\epsilon^Y}$) of the data model (**??**). The most popular estimates are computed by the method of maximum likelihood, whose implementation in SsfPack implicitly assumes a joint Gaussian distribution for the data. *SigDiagObj.ox* uses SsfPack's estimation method, which maps the SARIMA model (**??**) into State Space Form, and maximizes the likelihood using state space methods. This is numerically efficient and stable, and is to be preferred over more direct approaches. One limitation is that *SigDiagObj.ox* will not select model orders for the user – one must know ahead of time what SARIMA model orders are desired. Essentially, the user specifies the differencing orders (d, D) and a collection of initial parameter values, which implicitly determine the SARIMA model orders (p, P, q, Q) for the autoregressive and moving average polynomials.

One is also able to fix all of the parameters with user-selected values. This facility is useful for testing *a priori* defined filters (i.e., model-based filters whose parameters are not determined by the data, but rather by the user) on the data. One snag of the Hillmer-Tiao method is that some SARIMA models are inadmissable in the sense that they do not possess a decomposition. By overwriting certain parameter values, the user may be able to impose an admissible model on the data.

Example Suppose we specify a $SARIMA(1,1,1)(0,1,1)_{12}$ model for the logged shoe series. We obtain the following data model via estimation:

$$(1-.15B)(1-B)(1-B^{12})Y_t = (1-.67B)(1-.35B^{12})\epsilon_t^Y, \qquad \sigma^2_{\epsilon^Y} = .00095$$

This is accomplished through calling *InitParams* $(< .6 >, <>, < -.6 >, < -.5 >, 0, < 1, 1 >)$ – which specifies the model and initializes the parameters – and *Estimate*(), which uses maximum likelihood estimation to produce parameter estimates. More detailed information on *InitParams* is provided in its documentation within *SigDiagObj.ox*.

3 Decomposition

We first stipulate component models according to (**??**), a process that involves assigning all the "left-hand" operators of (**??**) to the left-hand sides of the component models. These left-hand operators include the AR operators and the differencing operators; for the latter, it is important that the factors are distributed uniquely, so that the differencing polynomials for each component are relatively prime. This is a prerequisite of the model-based signal extraction theory – see McElroy (2005a). The MA operators for the component models are then determined according to the canonical decomposition method of Hillmer and Tiao (1982), implemented through the Ox routine *gendecomp.ox* of Aston and Koopman (2003).

The *gendecomp.ox* function operates in the same manner as the code in SEATS for producing canonical decompositions, although our implementation is somewhat more simplistic. By associating all nonseasonal AR operators with the trend, we essentially define a trend-cycle component, which is more simply referred to as the trend. In contrast, SEATS will often generate a fourth cycle component that is an ARMA process, with certain of the nonseasonal AR factors associated to it. We have chosen to lump the trend and cycle together, which seems to be a reasonable approach given our principal objective of assessing the quality of seasonal adjustments. If one is further interested in cyclical behavior, a four-component decomposition could be attempted, though this may encounter problems of admissability.

Example In the shoe series, the $AR(1)$ operator is associated with the trend:

$$\begin{aligned} U(B)S_t &= (1 + 1.26B + 1.06B^2 + .84B^3 + .58B^4 + .35B^5 + .15B^6 \\ &\qquad - .03B^7 - .14B^8 - .26B^9 - .25B^{10} - .42B^{11})\epsilon_t^S, \qquad \sigma^2_{\epsilon^S} = .00012 \\ (1-.15B)(1-B)^2 T_t &= (1 - .46B - .96B^2 + .49B^3)\epsilon_t^T, \qquad \sigma^2_{\epsilon^T} = .000053 \\ I_t &= \epsilon_t^I, \qquad \sigma^2_{\epsilon^I} = .00023 \end{aligned}$$

The above results were produced by calling the *Decompose*() function; the *PrintComponents*() function may be used to output the models to the screen.

4 Signal Extraction

We adopt the model-based, finite sample approach to signal extraction. The various estimates can be produced through a state space smoother, implemented for example in SsfPack, but we produce the filter matrix directly utilizing formulas in McElroy (2005a). This is necessary for our subsequent calculations of the diagnostics themselves – see Findley et al. (2004). For each of the three components – trend, seasonal, and irregular – we produce the appropriate signal extraction matrix, each of whose rows corresponds to the time-dependent filter for the corresponding time point. In other words, the ith row of the filter matrix F consists of the filter coefficients that, when applied to the data Y_t, produce an estimate of the signal at time i. For example, if estimating the seasonal we have

$$FY = \hat{S} = \mathbb{E}[S|Y],$$

where $Y = (Y_1, Y_2, \cdots, Y_n)^{'}$ and $S = (S_1, S_2, \cdots, S_n)^{'}$ is the seasonal. In this stage of the program, we produce the filter matrix F and the signal estimate $\hat{S}$.

There are six quantities considered as possible signals of interest. These represent appropriately differenced components or combinations of such as follows:

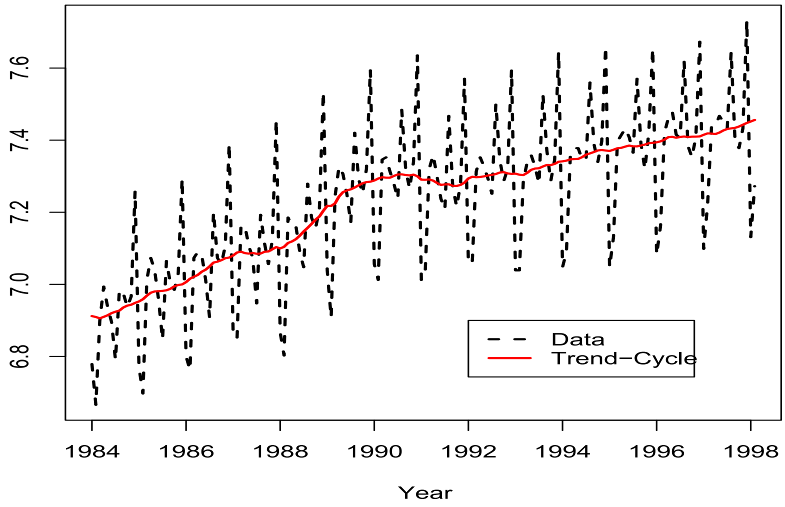
$$U(B)^D S_t, \qquad (1-B)^{d+D} T_t, \qquad I_t$$
$$U(B)^D (S_t + I_t), \qquad (1-B)^{d+D} (T_t + I_t), \qquad (1-B)^d (1-B^{12})^D (S_t + T_t)$$

The basic intra-component diagnostics are sample autocovariances – computed at various lags – of the differenced signal estimates, which means that we first estimate the differenced signals $U(B)^D S_t$, $(1-B)^{d+D} T_t$, etc., and then compute the lagged average of squares. If u denotes a vector of estimates for a differenced signal, then $u'u/n$ gives the lag zero sample autocovariance, assuming mean zero. If we let L be a lag matrix (of dimension equal to the length of u) with L_{ij} equal to 1 if $i = j + 1$ and 0 otherwise, then $u' L^h u/n$ is a quadratic form that yields the lag h estimate of the autocovariance (h is a non-negative integer).

The inter-component diagnostics are sample cross-covariances, computed for six pairings of components: S, I; S, T; S, TI; T, I; T, SI; and TS, I. We have a similar calculation, but now it involves two vectors u and v, containing estimates for two diverse differenced signals. An additional complication is that u and v could have different lengths (e.g., the irregular has length n but differenced trend has length $n - (d + D)$). Based on theoretical considerations, it is appropriate to trim the longer vector of its *first* values such that the two vectors have the same length; call this $\tilde{v}$ if v was the longer vector. Then $u'\tilde{v}/n$ gives the cross-covariance estimate.

Example In order to compute the signal extraction filters, we first call *buildDiffMatrices*() and then *buildCovMatrices*(), which construct the differencing and covariance matrices that will be needed. Then *ExtractSignals*() computes the filters and applies them to the data. Below in Figure 1 is a picture of the trend-cycle estimate in the shoe series together with the data.

Figure 1: Logarithm of U.S. Retail Sales of Shoe Stores, with Trend-Cycle estimate



5 Diagnostic Calculation

The diagnostics mentioned above provide a measure of variation in an estimated signal. Under the Null Hypothesis that our model for Y_t is correct, we can compute the mean and variance of this diagnostic and thereby construct a standardized diagnostic. Further details are developed in McElroy (2005b); here we discuss the interpretation and application of the diagnostics.

The standardized diagnostic is asymptotically normal under some mild conditions, and hence large positive or negative values indicate rejection of the specified model, whether this was generated through estimation or *a priori* methods. Since some of the diagnostics may be significant while others are not, the extreme values indicate model inadequacy only in certain portions of the data's spectrum. Roughly speaking, a significant diagnostic indicates that the spectral density of the differenced data $W_t = (1-B)^d(1-B^{12})^D Y_t$ is poorly modelled at the frequencies corresponding to the component of interest. For example, an extreme value of the trend diagnostic indicates poor modelling (and hence filtering) of the low frequencies, whereas an extreme value of the seasonal diagnostic would show poor modelling of the six seasonal frequencies. This interpretation is not completely rigorous – the diagnostics need to be further modified for that to happen. It is also difficult to interpret positive versus negative values of the diagnostic; it is safer to stick to the two-sided alternative. See McElroy (2005b) for a discussion.

The final output of the program includes the p-values for a two-sided test along with the standardized diagnostics, for each of the six signals defined above. Significant p-values indicate that adjustment to the model may be needed to improve signal extraction.

Example We call the function *ComputeLagDiagnostics*(h) with $h = 0, 1, 12$ and obtain the following output:

Table 1. Auto-covariance Diagnostics with p-values

Signal	Lag Zero	p-value	Lag One	p-value	Lag Twelve	p-value
Irregular	-0.423	.336	0.226	.411	0.648	.258
Trend	0.088	.465	-1.168	.121	0.631	.264
Seasonal	-0.043	.483	-0.058	.477	0.214	.415
Trend-Irregular	-0.467	.320	0.697	.243	0.458	.324
Seasonal-Irregular	-0.082	.467	0.019	.492	0.101	.460
Trend-Seasonal	-0.213	.416	0.450	.326	-0.171	.432

These diagnostics indicate adequacy of the filters, since none of the p-values are significant. For cross-covariance, we call the function *ComputeCrossDiagnostics*() and obtain:

Table 2. Cross-covariance Diagnostics with p-values

Signal Pair	Diagnostic	p-value
Seasonal, Irregular	0.637	.262
Seasonal, Trend	-0.853	.197
Seasonal, Trend-Irregular	-1.412	.079
Trend, Irregular	1.067	.143
Trend, Seasonal-Irregular	-0.345	.365
Trend-Seasonal, Irregular	0.738	.230

Again, the high p-values indicate model adequacy with respect to filtration.

No-Ends In Findley, McElroy, and Wills (2004), a further adjustment to the diagnostic is suggested, namely to trim the beginning and ends of each vector of estimates u, and adjust the statistical normalization appropriately. In that paper, it was proposed that this "no-ends" version would eliminate irritating end effects that interfered with the finite sample performance of the diagnostic. *SigDiagObj.ox* can also produce the "no-ends" version of the above diagnostics (not shown here).

6 Conclusion

The program *SigDiagObj.ox* performs SARIMA model estimation, canonical decomposition, signal extraction, and diagnostic calculation, given a model specification. Current implementation assumes a decomposition into seasonal, trend-cycle, and irregular; we hope to generalize this to four components if practicable. Also, the constraints on the differencing orders d and D make the current program easier to apply; future work will focus on addressing these constraints in the interest of making the program more broadly applicable. The incorporation of these diagnostics into *X–13–ARIMA–SEATS* will allow their testing upon hundreds of series at once, facilitating a large-scale empirical study. Also under development is a version of *SigDiagObj.ox* that allows for a separate "transient" component, which can model a separate cycle or a sampling error component, for example. This new version, called *Hybrid.ox*, was designed to be an extension of the diagnostics to handle series with sampling error, such as those encountered at the Bureau of Labor Statistics.

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