The Use of Signal Filtering for Hog Inventory Estimation

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Abstract

The National Agricultural Statistics Service (NASS) uses probability surveys of hog owners to estimate quarterly hog inventories in the United States at the national and state levels. NASS receives data from external sources on the number of Canadian hog imports and exports; Canadian feeder pigs; and farm and commercial slaughter counts. A panel of commodity experts which forms the Agricultural Statistics Board (ASB) reviews the proprietary survey results and industry transaction data and compares them against a set of inter-inventory relationship constraints. Given the internal survey data, external transaction data, and the set of inventory relationship constraints, the ASB establishes the NASS official published hog inventory estimates for the estimation quarter. The goal of this paper is to propose the estimation of hog inventories by combining the NASS proprietary survey results, the non-proprietary hog transaction data, the ASB panel expert analysis, and the inter-inventory relationship constraints using statistically defensible methodology. In order to achieve this goal, this paper demonstrates the expression of hog inventories in State-Space representation for use with an Extended Kalman Filter. Allocation of the U.S. level inventory estimates to the state level is formulated using Restricted Least Squares theory.

Hog Estimation Overview

The current process used in hog inventory estimation has been stationary for many decades. The sheer length of this probationary period leads to the question – why change the process now? The answer to this question requires a clear grasp of the scope of hog inventory estimation. To this end, the paper is structured to introduce fundamental concepts that provide a necessary foundation for understanding the current hog estimation process. This includes descriptions of the full spectrum of hog inventory items; background on the survey design and types of survey estimates; details about the non-proprietary inventory transaction data and its sources, and some explanation as to why the data provides a highly influential role in hog inventory estimation; a breakdown of the inter-inventory relationship constraints and their role in hog inventory estimation; information on the ASB and its origin and function in hog inventory estimation; and lastly, a brief introduction to key Office of Management and Budget (OMB) standards and guidelines for survey estimation. Once the fundamental details of hog inventory estimation have been conveyed, the paper will provide a brief overview of State-Space representation and the system equations relevant to hog inventory estimation. Following the overview of the State-Space system equations, the paper will then derive the system equations that express hog inventories in State-Space form. Hog inventories expressed in State-Space form will be used in conjunction with the Extended Kalman Filter in order to estimate those inventories given the survey results, non-proprietary inventory transaction data, inter-inventory relationship constraints, and the ASB analysis. The paper will then cover hog inventory estimation at the U.S. and state levels followed by a comparison of empirical results calculated from three different parameterizations of the hog inventory system equations. The three different parameterizations or “treatments” pertain to various ways of handling the ASB expert analysis and its role in the estimation of hog inventories.

1 Published Hog Inventory Items

This section provides an overview of the scope of published hog inventory items. It describes the level of detail at which inventories are published and how they relate.

The National Agricultural Statistics Service (NASS) publishes quarterly hog inventory estimates in terms of the number of hogs living on hog operations in a domain of reference at the end of that quarter. Hog owners, including contractors, are the target population. The quarters estimated are March, June, September, and December. The interpretation of a March inventory count means the number of hogs living on hog operations for the corresponding domain on March 1st. Likewise, a June inventory count refers to the number of hogs on June 1st. For a given domain and quarter of reference, hog inventories are provided for ten categories of inventory. The first of these categories is the sum total of all hogs and pigs. The total number of hogs and pigs is also partitioned into market weight group classes. The first of these market weight group classes is the number of market hogs weighing less than 50 lbs. The second group is those market hogs between 50 and 119 lbs. The third and fourth market weight groups are comprised of market hogs between 120 and 179 lbs, and market hogs over 180 lbs, respectively. The sum total of these four weight classes is reported as total market hogs. Additional categories cover hog reproduction and include the number births which survive weaning, the number of sows farrowed, and breeding herd size. The ratio of pig crop (weaned births) to sows farrowed is reported as the litter rate and can be interpreted as the mean number of pigs which survive past weaning born to a sow. The sum of the four weight classes equals the total number of market hogs, and the sum of total market hogs plus breeding herd equals the total number of hogs and pigs. Pig crop is contained within market hogs less than 50 lbs and market hogs 50-119 lbs. Sows farrowed are contained within breeding herd. In the estimation quarter of December 2009, the first two weight groups were redefined to the present day definitions. The first weight class of market hogs less than 50 lbs had been previously reported as market hogs less than 60 lbs, and the second weight class of market hogs between 50 and 119 lbs had been previously reported as market hogs between 60 and 119 lbs. Table 1 contains a summary list of the hog inventory items published at the U.S. and state levels in the National Agricultural Statistics Service's quarterly Hog Report[1].

Table 1

Item Number	Inventory Item	Notation	Relationship
1	Total Hogs and Pigs	H	
2	Pig Crop (weaned births)	P	
3	Sows Farrowed	S	
4	Market Hogs less than 50 lbs	G_1	
5	Market Hogs 50 – 119 lbs	G_2	
6	Market Hogs 120 – 179 lbs	G_3	
7	Market hogs greater than 180 lbs	G_4	
8	Market Hogs	M	$\sum_{k=1}^{4} G_k$
9	Breeding Herd	B	$H - M$
10	Litter Rate	T	$\frac{P}{S}$

2 Hog Survey Measurements

This section covers the hog inventory survey including sampling frames, design, and types of estimators.

Survey estimates of inventory are calculated from a stratified simple random sample design. The strata are partitioned according to size of hog operation with respect to the number of total hogs and pigs stored in NASS's list frame. The sampling unit is any hog operation with the capacity to raise breeding or market hogs. In addition to the hog operation population list frame estimate referred to as the ADXX list frame survey estimate, there is also a multiframe estimate (ADMW). The ADMW estimate contains the inference for the hog operation population list frame plus an area frame component. The area frame component estimates the number of hogs belonging to owners who are Not On the List (NOL) frame. This NOL component is estimated using a separate area frame sampling design. The area frame survey is conducted on an annual frequency; however, the results are used to calculate quarterly estimates of the NOL component for total hogs and pigs, pig crop, sows farrowed, and breeding herd. The market weight group multiframe estimates are calculated on an annual basis in the quarter of December, and the other quarters are adjusted based on the December ADXX and NOL ratio. The NOL component adjusts for undercoverage of the list frame component. U.S. and state-level estimates of the variance of the survey estimates are also calculated according to the sampling design. Historical plots comparing NASS official ASB estimates and the ADXX and ADMW survey results are shown in Figure 1 through Figure 10.

Figure 1

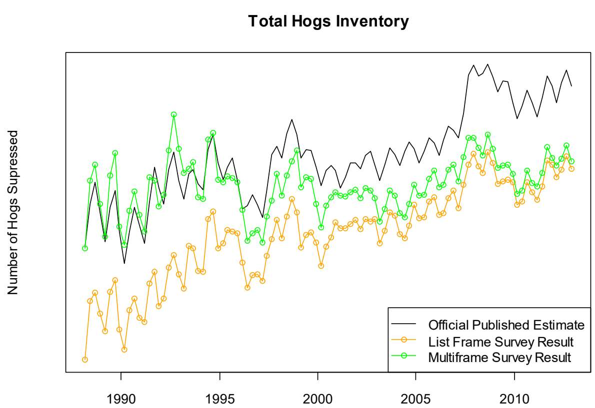


Figure 2

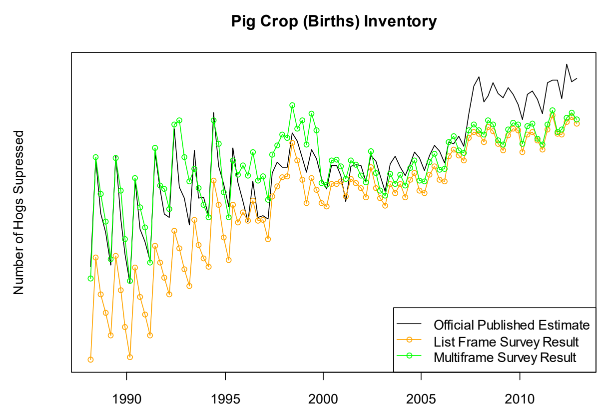


Figure 3

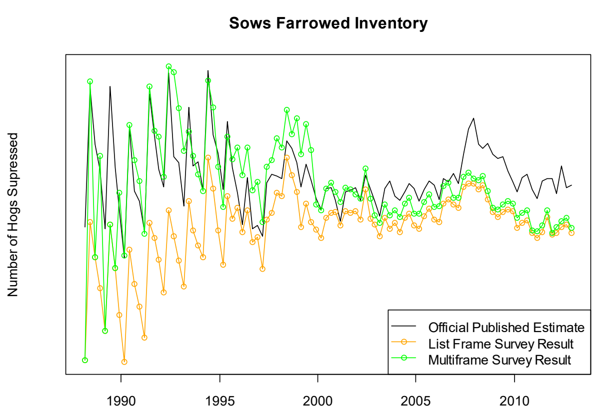


Figure 4

Figure 5

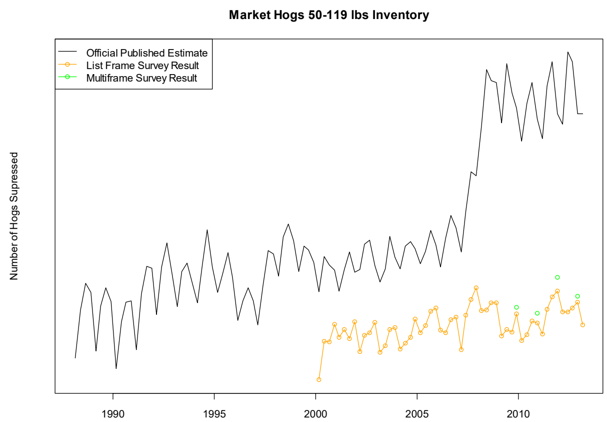


Figure 6

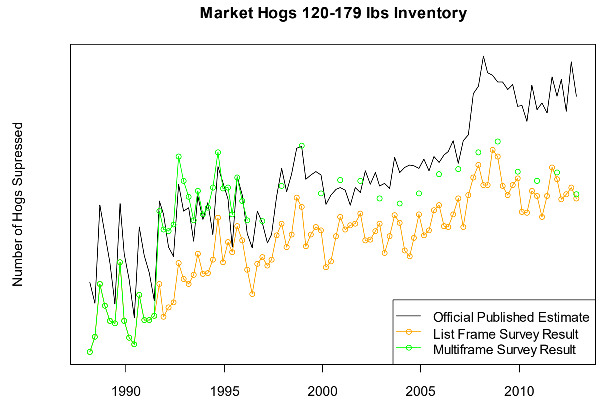


Figure 7

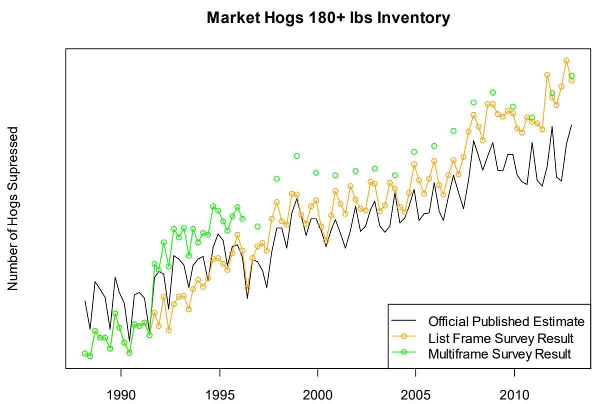


Figure 8

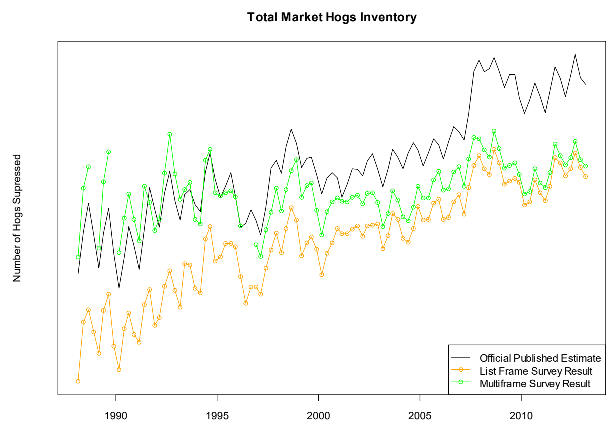


Figure 9

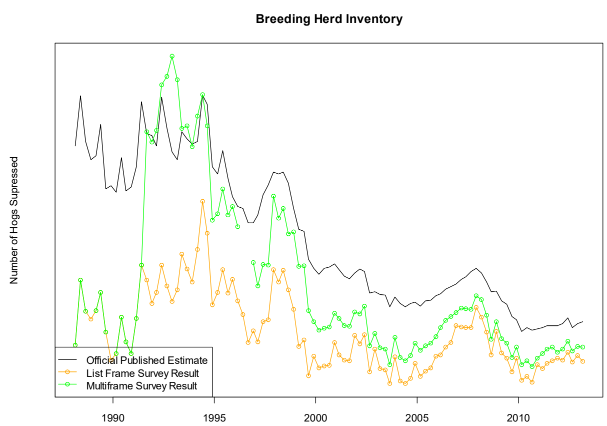
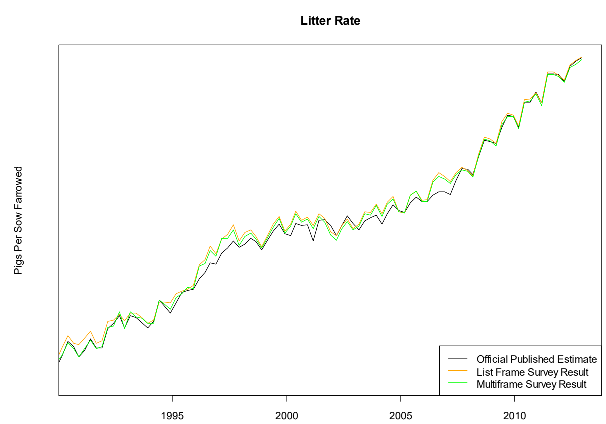


Figure 10



For each of the graphs, the differences between survey results and published inventory are attributed to the additional information provided by the non-proprietary inventory transaction data and a set of assumptions in the form of constraints on how inventory items relate one to another. These assumptions build the foundation for the system equations of the signal filter that will be derived in this paper. Actual survey results and their variances are never released to the public.

3 Non-proprietary Inventory Transaction Data

This section provides details on the inventory transaction data referred to in the previous section which NASS obtains from external sources. This data is highly influential in the differences between the survey results and the NASS official published estimates determined by the ASB. The transaction data plays a key role in inter-inventory relationship constraints that will be introduced in the next section.

The life of a hog from birth to slaughter is approximately six months. This implies that the reported pig crop in a given quarter is reflected in slaughter estimates two quarters later. Slaughter data is obtained from the Agricultural Marketing Service (AMS) and is partition into farm slaughter, federally inspected commercial slaughter, and non-federally inspected commercial slaughter. According to AMS data, federally inspected commercial slaughter amounts to roughly 99% of total slaughter. This is significant because the majority of slaughter is federally inspected. Although NASS does not receive variance estimates for any of the hog commodity transaction data from outside sources, the slaughter estimates are assumed to have very low variance due to federal inspection. In addition to hog slaughter, NASS receives data on hog imports and exports to and from Canada from the Department of Commerce through the Foreign Agricultural Service. The transaction data is available at the U.S. level only. Table 2 lists the inventory transaction data and the notation required to formulate the hog inventory constraints which will be given in section 4.

Table 2

Item Number	Data	Notation	Function
1	Slaughter	L	
2	Imports	I	
3	Exports	E	
4	Canadian Feeder Pigs	C	
5	Death Loss	D	
6	Balance Sheet Net	BSN	$I - E - D - L$

4 Hog Inventory Relationship Constraints

This section introduces the inter-inventory relationship constraints. These constraints are mathematical expressions which relate inventory items to each other and to the external transaction data. The survey results are not published because they do not satisfy these constraints.

The justification for publishing inventory estimates other than the survey results is the satisfaction of a set of assumed constraints. These constraints relate current inventory to past inventory, relate current and past inventory to the external transaction data, and reflect the hog growth cycle. The survey results do not satisfy the constraints. True hog and pig inventories are assumed to satisfy the constraints introduced in this section. The constraints will be given as mathematical expressions using the notation from Table 1 and Table 2. The subscript t will be used to index time, where the interval between t and $t + k$ represents k quarters, or k consecutive three month intervals. The quarter of reference for $t = 1$ refers to the first quarter, and the quarter of reference $t = n$ refers to the last quarter in the time series. In general, $t = n$ refers to the most recent quarter of inventory estimates. All constraints pertain to the U.S. level of estimation.

4.1 The Balance Sheet Equation

Total hog inventory can be compared to a deposit account; an accounting system composed of deposits and withdrawals. Deposits are analogous to increases in inventory such as births and imports. Withdrawals are the decreases in inventory which consist of slaughter, exports, and death loss. This balance sheet concept forms the balance sheet equation where one quarter's total hogs and pigs inventory is equal to the previous quarter's total hogs and pigs inventory plus the quarter's deposits minus the quarter's withdrawals. The balance sheet equation is then

$$\begin{aligned} H_t &= H_{t-1} + P_t + I_t - E_t - D_t - L_t \\ &= H_{t-1} + P_t + BSN_t \end{aligned}$$

Let BSR_t be defined as the balance sheet residual at time t and

$$BSR_t = H_t - H_{t-1} - P_t - BSN_t$$

If $BSR_t = 0$ then $H_t = H_{t-1} + P_t + BSN_t$ and the system is in balance. In setting published total hogs and pigs inventory, the balance sheet residual is allowed to vary at most by approximately one day's slaughter (approximately 500,000 hogs). It then follows that

$$\begin{aligned} |BSR_t| &\leq 500{,}000 \\ |H_t - H_{t-1} - P_t - BSN_t| &\leq 500{,}000 \end{aligned}$$

We will now define the three month, six month, and twelve month balance sheet constraints as

$$\left| \sum_{k=0}^{K} H_{t-k} - H_{t-1-k} - P_{t-k} - BSN_{t-k} \right| \leq 500{,}000 \tag{1}$$

with $K = 0$, $K = 1$, and $K = 3$, respectively.

Figure 11

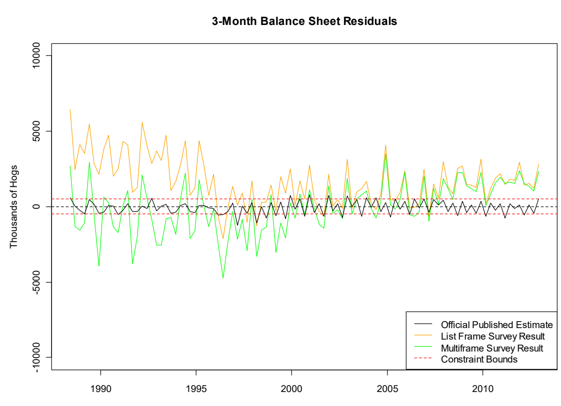


Figure 12

Figure 13

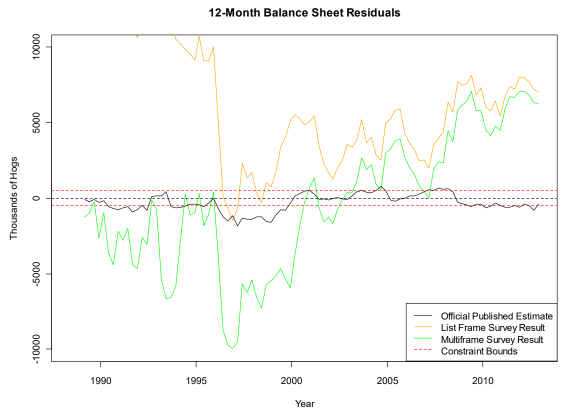


Figure 11, Figure 12, and Figure 13 plot the 3-month, 6-month, and 12-month balance sheet residuals for the ADXX (list frame) and ADMW (multiframe) survey results. The balance sheet residuals of the published ASB estimates are included for comparison with those of the survey results. The plots demonstrate the ASB's attempt to contain the balance sheet residuals of the published estimates within the 500 thousand hogs bounds.

4.2 Death Loss Ratio

The concept of the Death Loss Ratio is to acknowledge that there is a quantity of pig crop that dies and therefore cannot be counted in the market weight groups. These pigs survive past weaning and are within scope of the definition of pig crop. The weight of pigs born during a quarter is distributed between the first and a proportion of the second weight group. We will call that proportion α. If we look at these concepts in terms of annual increases, we have

$$\frac{P_t + C_t}{P_{t-4} + C_{t-4}} > \frac{G_{1_t} + \alpha_t G_{2_t}}{G_{1_{t-4}} + \alpha_{t-4} G_{2_{t-4}}} \quad (2)$$

Canadian Feeder Pigs are grouped with the births. Conceptually, this expression conveys that the annual increase in the number of pigs born and are weaned is greater than the annual increase in the first two market weight groups. The inequality implies disappearance from the weight group increase due to death loss after weaning that quarter. The value for α_t is time dependent due to a change in definition of the first two weight groups which happened in December 2009. Prior to December 2009, weight group 1 consisted of those market hogs weighing less than 60 lbs. Weight group 2 was composed of those market hogs weighing between 60 and 119 lbs. The parameter α_t is evaluated as follows:

$$\alpha_t = \begin{cases} 0.33 & t \text{ is prior to 2008} \\ 0.42 & \text{otherwise} \end{cases}$$

The values for alpha were determined by commodity analysts. The current commodity analysts enforce bounds which give the following Death Loss "Difference" constraint:

$$0.0041 \le \frac{P_t + C_t}{P_{t-4} + C_{t-4}} - \frac{G_{1_t} + \alpha_t G_{2_t}}{G_{1_{t-4}} + \alpha_{t-4} G_{2_{t-4}}} \le 0.0043 \quad (3)$$

The Death Loss Ratio constraint in equation (4) is the true ratio version of the Death Loss Difference in equation (3). Figure 14 plots the historical Death Loss Ratio of the published inventory items.

$$1.0041 \lesssim \frac{P_t + C_t}{P_{t-4} + C_{t-4}} \left(\frac{G_{1_t} + \alpha_t G_{2_t}}{G_{1_{t-4}} + \alpha_{t-4} G_{2_{t-4}}} \right)^{-1} \lesssim 1.0043 \quad (4)$$

Figure 14

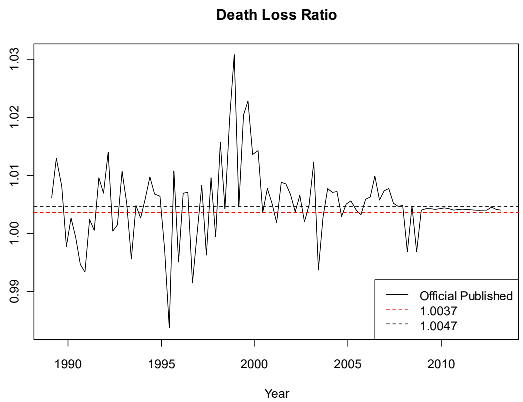


Figure 15 graphs the ASB death loss relationship by comparing the right side of equation (2) versus the left. Figure 16 graphs the left and the right sides of equation (2) using the ADXX survey result.

Figure 15

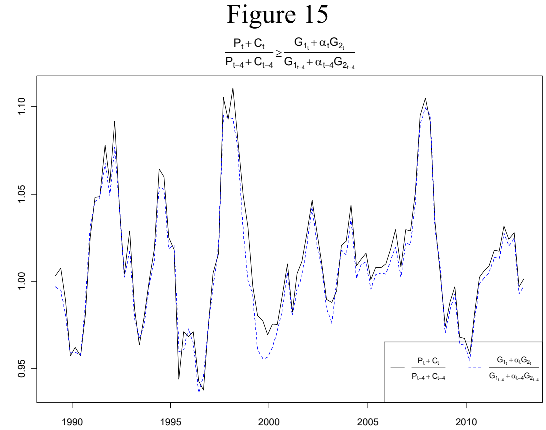
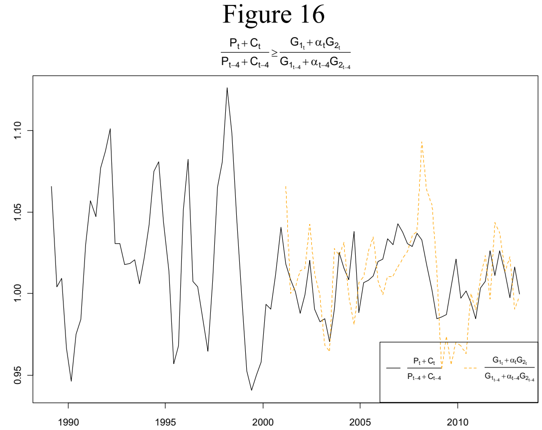


Figure 16



4.3 Weight Group Transition

Where the Death Loss Ratio maps hog births to their corresponding weight classes during the quarter, the Weight Group Transition constraint maps those births and their weights to the heavier weight groups the following quarter. The Weight Group Transition constraint is an assumption about the growth of pigs within weight classes. It links the lighter two weight classes to the heavier two weight classes over the passing of a quarter.

$$\frac{(1-\alpha_t)G_{2_t} + G_{3_t} + G_{4_t}}{(1-\alpha_{t-4})G_{2_{t-4}} + G_{3_{t-4}} + G_{4_{t-4}}} \ge \frac{G_{1_{t-1}} + \alpha_{t-1} G_{2_{t-1}}}{G_{1_{t-5}} + \alpha_{t-5} G_{2_{t-5}}} \quad (5)$$

This constraint implies that the annual increase in weight groups three, four and a proportion of the second is the annual increase in weight group one and a proportion of the second one quarter in the past.

Figure 17

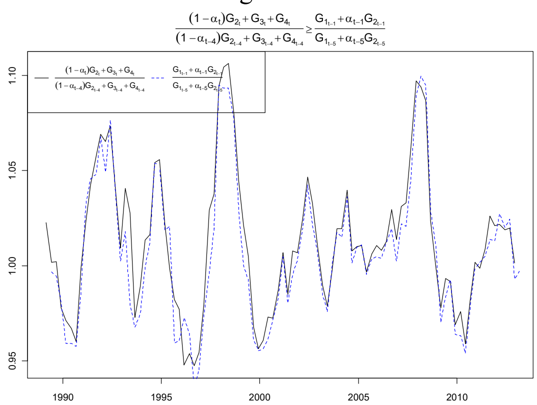


Figure 18

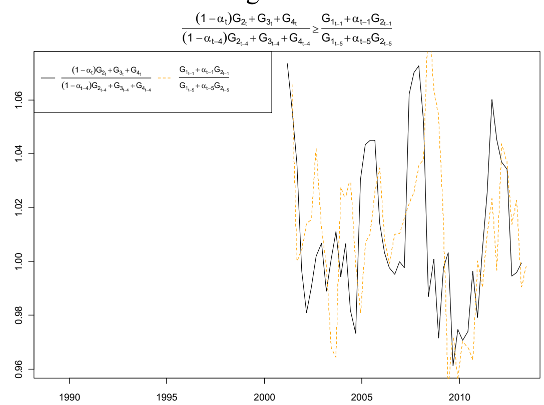


Figure 17 plots the right side and left side of the inequality (5) using the ASB values. The right side of inequality (5) is the lagged version of the right side of equation (2). Figure 18 plots the right side and left side of the inequality (5) using the list frame ADXX values. The Death Loss Ratio constraint and the Weight Group Transition constraint represent the flow of hogs from pig crop births through the market weight groups until slaughtered.

4.4 Pig Crop – Slaughter Ratio

The time between birth and slaughter for a pig is approximately six months or two quarters. This implies that hogs born in quarter t are slaughtered in quarter $t+2$. The commodity analysts translate this concept into a ratio constraint where the annual increase in slaughter is equivalent to the increase in births two quarters in the past. This constraint is formulated as

$$\frac{L_t}{L_{t-4}} = \frac{P_{t-2}}{P_{t-6}} \tag{6}$$

Figure 19

Figure 19 graphs the left side slaughter ratio versus the right side of equation (6) with substituted ASB values, and Figure 20 adds substituted survey values for the right side of equation (6) for comparison of the survey performance versus the ASB published estimates.

4.5 Market Hogs – Slaughter Ratio

Constraint 4.4 is extended to include all market hogs by the annual increase in six months of slaughter.

$$\frac{L_t + L_{t-1}}{L_{t-4} + L_{t-5}} = \frac{\sum_{i=1}^{4} G_{i_{t-2}}}{\sum_{j=1}^{4} G_{j_{t-6}}} = \frac{M_{t-2}}{M_{t-6}} \tag{7}$$

Equation (**7**) encompasses all four weight classes and conveys that the annual increase in total hogs with the exception of those hogs reserved for breeding is essentially the annual incease in two quarters of slaughter. Figure 21 plots the left and right sides of equation (7) in terms of the ASB published values. Figure 22 substitutes the ADXX list frame survey results into the right side of equation (7)

Figure 23

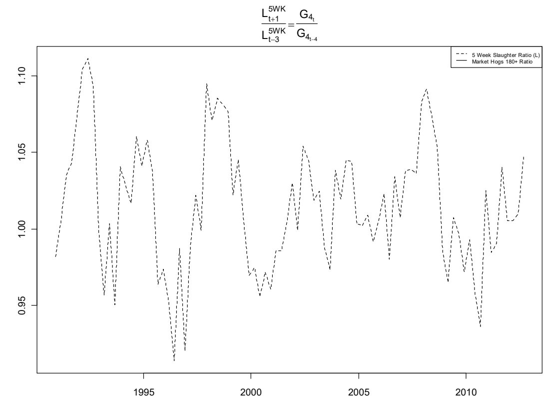


Figure 24

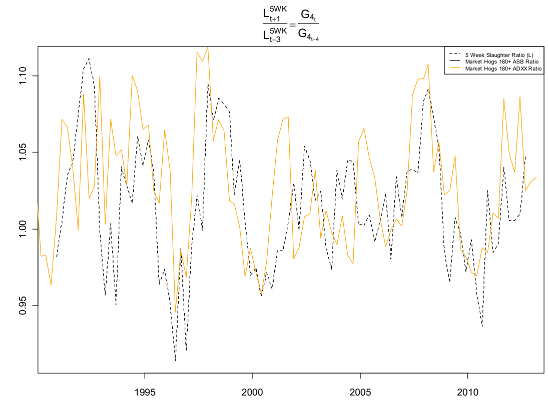


Figure 23 compares the left and right sides of equation (8) substituting ASB values. Figure 24 substitutes the list frame ADXX survey result into the right side of equation (8) *survey results described in section*

NASS surveys its sampling frame of hog operations, obtains non-proprietary hog inventory transaction data, and submits these data to the ASB panel with the purpose of establishing quarterly hog inventory estimates that make sense from an historical standpoint, reflect congruency with inventory transaction data, and maintain inter-inventory consonance. This paper will demonstrate that the goals of NASS and the ASB, and the goals of the OMB Standards and Guidelines can be achieved using an Extended Kalman Filter. The Kalman Filter is a signal filtering tool for which there is an abundance of literature, supporting its use as statistically defensible methodology. Durbin and Koopman write, "the object of filtering is to update our knowledge of the system each time a new observation is brought in (Durbin and Koopman 2012)." Filtering is a methodology which can combine all observations of hog inventory; including the survey measurements from section

8.2 The Observation Equation

The observation equation relates a set of measurements or observations of the *state* to the *state*. In this paper, "observations" and "measurements" will be used interchangeably. The linear observations are modeled by the relationship

$$y_t = A_t x_t + v_t \tag{12}$$

where y is a $k \times 1$ vector of measurements, A is a $k \times m$ measurement matrix which defines the linear relationship between the *state* x_t and the observations y_t, and v_t is a $k \times 1$ observation noise vector with a Gaussian distribution. The first and second moments are $E[v_t] = 0 \;\forall\; t$ and $E\left[v_t v'_{t-j}\right] = \begin{cases} R & j = 0 \\ 0 & j \neq 0 \end{cases}$. In addition, $E\left[w_{t-i} v'_{t-j}\right] = 0 \;\forall\; i, j$. If the relationship between the measurements and the *state* is nonlinear, equation (13) is used where $H(\cdot)$ represents the system of nonlinear measurement relationships as a function of the *state*.

$$y_t = H_t(x_t) + v_t \tag{13}$$

9 Hog Inventory Transition Equations

This section derives the hog inventory State-Space transition equations by putting equations (10) and (11) in terms of the published hog inventory items from Table 1 so that hog inventories can be estimated using the Extended Kalman Filter. This will be done using the constraints listed in section 4.

In order to estimate hog inventories using a Kalman Filter, they must be formulated in State-Space representation. This involves defining the *state* vector from the published items in Table 1, determining the parametric transition relationships for each *state* vector element that conform to the transition equations (10) and (11), and determining the parametric relationships of all measurements and observations to the *state* which conform to the observation equations (12) and (13). The constraints listed in section 4 define the behavior rules that can be adapted into State-Space form which will be demonstrated in this section. Before establishing the *state* vector, we must define some weight group functions and a linear filter operator which will assist in formulating the transition relationships. We define the weight group functions f_1, f_2, f_3, and f_4 as

$$f_{1_t} = G_{1_t} + G_{2_t} \tag{14}$$

$$f_{2_t} = \ln\left(\frac{G_{1_t} + \alpha_t G_{2_t}}{P_t + C_t}\right) \tag{15}$$

$$f_{3_t} = \ln\left[(1 - \alpha_t) G_{2_t} + G_{3_t} + G_{4_t}\right] \tag{16}$$

$$f_{4_t} = \ln\left(G_{4_t}\right) \tag{17}$$

These functions should be familiar from the terms in the hog inventory constraints in section 4. We will use them to develop *state* transition relationships. In addition, we will use the linear filter operator of the first and fourth difference

$$\Delta(x_t) = x_t - x_{t-1} - x_{t-4} + x_{t-5} \tag{18}$$

Given the weight group function equations (14) - (17) and the linear filter equation (18), the hog inventory *state* vector elements and corresponding transitions are listed in Table **3** and Table **4**, respectively.

Table 3

Item	Element of $\boldsymbol{x_t}$	Values of k (x)
1	$\Delta(H_{t-k})$	$k \in \{0,1,2,3\}$
2	$\Delta(P_{t-k})$	$k \in \{0,1,2,3\}$
3	$\Delta(S_{t-k})$	$k \in \{0,1,2,3\}$
4	$\Delta\left(f_{1_{t-k}}\right)$	$k \in \{0,1,2,3\}$
5	H_{t-k}	$k \in \{1,2,3,4,5\}$
6	P_{t-k}	$k \in \{1,2,3,4,5\}$
7	S_{t-k}	$k \in \{1,2,3,4,5\}$
8	$f_{1_{t-k}}$	$k \in \{1,2,3,4,5\}$
9	$f_{2_{t-k}}$	$k \in \{0,1,2,3,4,5,6\}$
10	$f_{3_{t-k}}$	$k \in \{0,1,2,3,4,5,6\}$
11	$f_{4_{t-k}}$	$k \in \{0,1,2,3,4\}$
12	u_{x_t}	$x \in \{H, P, S, f_1, f_2, f_3, f_4\}$

Table 4

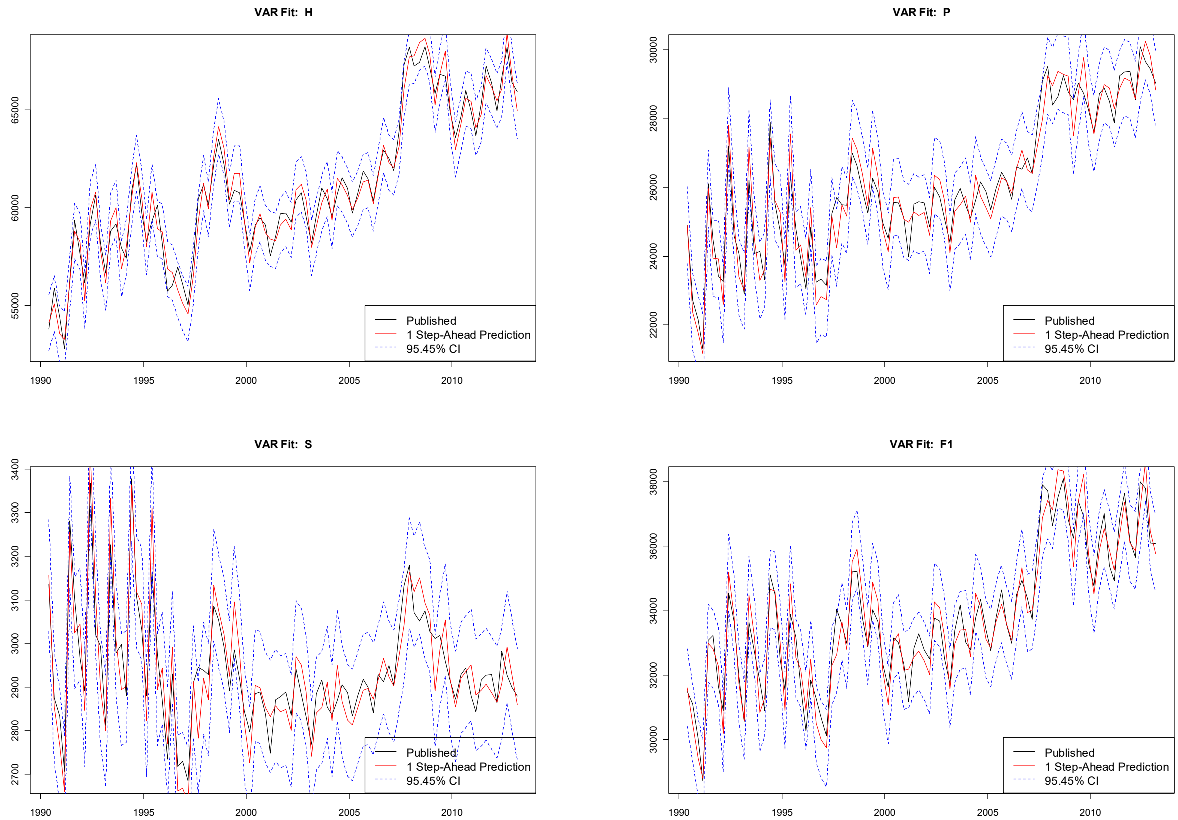
Item	Section	Element	Transition Function
1	9.1	$\Delta(H_t)$	$\begin{bmatrix}\Delta(H_t)\\ \Delta(P_t)\\ \Delta(S_t)\\ \Delta\left(f_{1_t}\right)\end{bmatrix} = \sum_{\mathrm{i}=1}^{4} \boldsymbol{\Phi}_{\mathbf{i}} \begin{bmatrix}\Delta(H_{t-i})\\ \Delta(P_{t-i})\\ \Delta(S_{t-i})\\ \Delta\left(f_{1_{t-i}}\right)\end{bmatrix}$
2	9.1	$\Delta(P_t)$	
3	9.1	$\Delta(S_t)$	
4	9.1	$\Delta\left(f_{1_t}\right)$	
5	9.2	f_{2_t}	$f_{2_t} = f_{2_{t-4}} - \ln(1.0042)$
6	9.3	f_{3_t}	$f_{3_t} = \ln\left[\frac{e^{f_{2_{t-1}}}(P_{t-1} + C_{t-1})}{e^{f_{2_{t-5}}}(P_{t-5} + C_{t-5})}\right] + f_{3_{t-4}}$
7	9.4	f_{4_t}	$f_{4_t} = \ln\left(f_{1_{t-1}}\right) - \ln\left(f_{1_{t-5}}\right) + f_{4_{t-4}}$
8	10.2	u_{x_t}	$u_{x_t} = u_{x_{t-1}}$

The additive process noise term w_t has been omitted for convenience from Table 4. In sections 9.1-9.4 we will develop justification for the transition relationships with the exception of Table 4 item 8 which is an estimate of the true survey bias which will be introduced in section 10.2. The true survey bias u of inventory item x defined as u_x must be introduced here because it will be estimated and therefore must be defined as part of the state and given a transition.

9.1 Hogs, Pig Crop, Sows Farrowed, and Weight Group Function $\boldsymbol{f_1}$

We transition total hogs, pig crop, sows farrowed, and f_1 using a Vector Autoregressive (VAR) model. Figure **26** illustrates the VAR transition model step-ahead predictions compared to ASB published numbers.

Figure 26



From visual inspection, the VAR model appears to be a reasonable transition. A 95.45% (two standard deviations of a Gaussian distribution) confidence interval is included to demonstrate the error term variance estimate. The VAR error term is analogous to the process noise in State-Space representation. For a simple comparison of the VAR prediction performance between inventory items, we can look at a bar plot of Coefficients of Variation calculated by the square-root of the VAR error term variance estimates divided by the series' means.

Figure 27

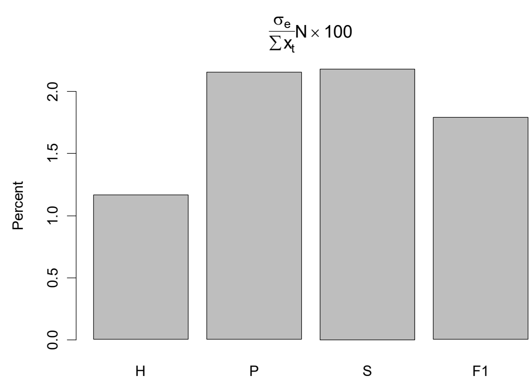


Figure 27 supports the general assumption that total hog and pigs is the "most stable" item to model at the U.S. level, meaning that the estimate of the standard error of the noise process is the smallest relative to the mean size for total hogs and pigs in comparison to the other inventory items' process noise standard errors and mean sizes. It should also be noted that the Vector Autoregressive order is fixed at 4. The order that provides the minimum for canonical fit statistics such as Akaike's Information Criterion (AIC) changes depending on the time window used to estimate the VAR model. Common AR orders of 3 and 4 as well as a couple outlier orders have been empirically produced. The order 4 was chosen because it was the mode and is more intuitive; an order of 4 will transition one year of data.

9.2 Weight Group Function f_2

Weight group function f_2 and its transition are derived from the Death Loss Ratio constraint in section 4.2. Defining λ_t as the death loss ratio at time t, we can rewrite the Death Loss Ratio as a true ratio (commodity analysts calculate it as a difference) in terms of the weight group function f_2.

$$\lambda_t = \left(\frac{P_t + C_t}{P_{t-4} + C_{t-4}}\right)\left(\frac{G_{1_t} + \alpha_t G_{2_t}}{G_{1_{t-4}} + \alpha_{t-4} G_{2_{t-4}}}\right)^{-1} \\ = e^{f_{2_{t-4}} - f_{2_t}} \tag{19}$$

The death loss ratio in equation (19) is expressed in terms of true hog and pig inventories. We can collect all terms of equation (19) at time $t-k|k=0$ on the left side of the equation and all terms at time $t-k|k>0$ on the right side of the equation which yields equation (20).

$$\frac{G_{1_t} + \alpha_t G_{2_t}}{P_t + C_t} = \frac{1}{\lambda_t} \frac{G_{1_{t-4}} + \alpha_{t-4} G_{2_{t-4}}}{P_{t-4} + C_{t-4}} e^{w_t} \tag{20}$$

We have added multiplicative process noise term e^{w_t}. By taking the natural log of both sides of equation (20), we can model a linear transition of f_2 with the process noise error term w_t in state space representation by

$$f_{2_t} = f_{2_{t-4}} - \ln(\lambda_t) + w_t \tag{21}$$

$$\ln(\lambda_t) = \ln(\lambda_{t-1}) + w_{\lambda_t} \tag{22}$$

Equation (21) incorporates the natural log of the death loss ratio as part of the *state* vector with its own random walk transition and process noise in equation (22). The death loss ratio is essentially formulated here as a time variant level in the *state* vector. This parameterization is more reflective of the true historical death loss ratio in Figure 14. However, to conform to the assumptions of the current commodity analysts, we will assume the death loss ratio λ_t is distributed symmetrically with constant mean value between its bounds of 1.0041 and 1.0043. Therefore, $E[\lambda_t] = 1.0042$. We replace λ_t in the f_2 transition with the fixed value of its expectation 1.0042 and the f_2 transition becomes

$$f_{2_t} = f_{2_{t-4}} - \ln(1.0042) + w_t \tag{23}$$

Figure 28 plots the official published ASB values for f_2 and the expectation of its transition in equation (23). It is more intuitive to examine the plot of $e^{f_{2t}}(P_t + C_t) = G_{1_t} + \alpha_t G_{2_t}$ in Figure 29

Figure 28

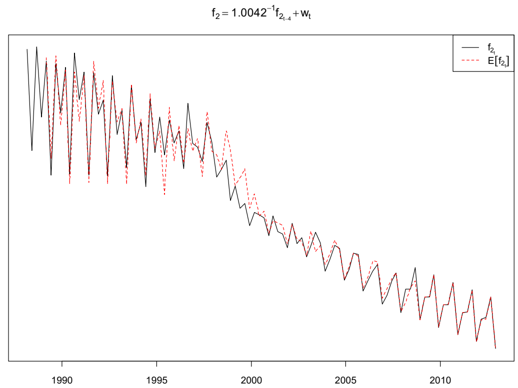
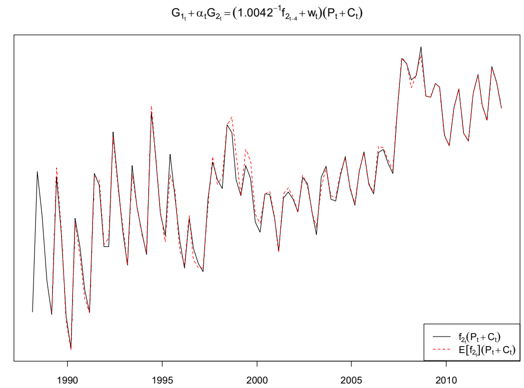


Figure 29



Canadian Feeder Pigs C_t is treated as a fixed parameter. The value is known as it is supplied as external inventory transaction data.

9.3 Weight Group Function f_3

The weight group function f_3 transition follows a similar derivation tof_2. The constraint 4.3 can be written in terms of f_2 and f_3.

$$\frac{e^{f_{3t}}}{e^{f_{3t-4}}} \geq \frac{e^{f_{2t-1}}(P_{t-1} + C_{t-1})}{e^{f_{2t-5}}(P_{t-5} + C_{t-5})} \tag{24}$$

Collection of terms at time $t - k|k = 0$ on the left and $t - k|k > 0$ on the right followed by taking the natural log of both sides of equation (24) yields the transition equation

$$f_{3_t} = \ln\left[\frac{e^{f_{2t-1}}(P_{t-1} + C_{t-1})}{e^{f_{2t-5}}(P_{t-5} + C_{t-5})}\right] + f_{3_{t-4}} + w_t \tag{25}$$

The model noise process is multiplicative with e^{w_t}. Figure 30 illustrates the official published ASB estimate of f_3 and its expected transition.

Figure 30

( )

9.4 Weight Group Function f_4

The transition for weight group 4 from Table 4 is not derived from any of the constraints from section 4.

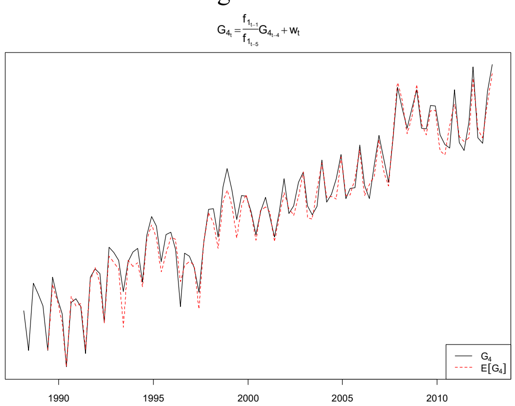
$$G_{4_t} = \frac{G_{1_{t-1}} + G_{2_{t-1}}}{G_{1_{t-5}} + G_{2_{t-5}}} G\_4_{t-4} \tag{26}$$

If we take the natural log of both sides of (26) with the multiplicative error term e^{w_t}, we have the transition equation

$$f_{4_t} = \ln(f_{1_{t-1}}) - \ln(f_{1_{t-5}}) + f_{4_{t-4}} + w_t \tag{27}$$

Its performance is illustrated below in Figure 31.

Figure 31



10 The Hog Inventory Observation/Measurement Equations

This section derives the hog inventory State-Space observation equations by putting equations (12) and (13) in terms of the published hog inventory items from Table 1 so that hog inventories can be estimated using the Extended Kalman Filter. This will be done using the constraints listed in section 4.

In order to put the measurements of inventory into the observation system of equations, we group the observations related to hog inventories into three categories according to source. We are using signal filtering methodology to estimate an unobserved signal for which we have "noisy" measurements. These measurements include the published ASB estimates, the survey results, and the non-proprietary hog inventory transaction data. We will categorize the ASB measurements of inventory as "expert analysis" measurements. The three categories of hog inventory observations are therefore expert analysis measurements, survey measurements, and external inventory transaction data. The categorization into these three groups helps distinctly separate how the types of measurements are treated. The relationships between inventory transaction data and hog inventories are defined in the observation equations through the relationship constraints introduced in section 4. Farm and commercial slaughter, hog imports and exports from Canada, and death loss estimates comprise this group. The survey results are treated as biased measurements of true inventories. The question remains how to treat the expert analysis measurements relative to true inventory. Sections 10.1 - 10.3 address the parameterization of each observation category.

10.1 Expert Analysis Estimates and True Inventory

The current hog inventory estimation process involves a panel of experts that take all given hog inventory data from internal surveys and external sources in order to find a solution that satisfies a set of assumptions in the form of constraints. This paper does not argue the validity or appropriateness of the constraints; rather, it proposes to incorporate the current process into a signal filtering model analog of the current hog estimation process. In order to maintain a smooth transition from the published board inventory estimates to published signal filter model estimates, the historical published ASB estimates must be included in the observation vector. Assuming that hog inventories will be estimated indefinitely, that particular observation or measurement must be continued. An example of methodology that eliminates ASB measurements and at the same time continues from where the ASB ends is to include a measurement in the observation vector that is the published measurement. The published inventory measurements are the ASB measurements before implementation of signal filtering methodology, and the model estimates become the published measurements post implementation of the Filter. This parameterization does require that certain complications be addressed, such as published revisions as new slaughter data and other data become available. In addition, model output becomes an observation, which is model input. An alternative approach is to continue using the ASB measurement as an "expert analysis" measurement. The previously published inventory becomes the past expert analysis measurements in the observation vector. Instead of publishing that ASB measurement, it is put into the signal filter with the other survey measurements and observation data. The Kalman Filter estimates are published together with their standard errors in the official release. Sections 10.1.1, 10.1.2, and 10.1.3 discuss various treatments and parameterizations of the ASB expert analysis measurements.

10.1.1 ASB Expert Analysis as an Unbiased Observation

Incorporating Signal Filtering methodology as an analogue of the current estimation process implies continuing to supply an ASB measurement. There are many ways this can be done. The inclusion of ASB inventory measurements allows expert opinion the possibility to exert some influence in a statistically defensible model. One concern is the possibility that expert opinion will exert too much influence on the filter estimates relative to the other measurements so that filter estimates are simply "perturbed" ASB estimates. This concern will be addressed in later sections with model results for various proposed parameterizations.

In order to establish a relationship between true inventory and measurements of true inventory, we must make some assumptions about true inventory. The ASB assumes that the survey results are biased and therefore publishes its own estimates of inventory rather than the survey results. Let us first assume that the ASB expert analysis estimates of true inventory are unbiased estimates of inventory. This is parameterized as

$$x_t^{ASB} = x_t + v_t \tag{28}$$

where $x \in \{H, P, S\}$. Another implication of this parameterization is $var(x_t^{ASB}) = R$. The variance estimate of the observation noise is parameterized as an estimate of the variance of the ASB estimate. Equation (28) is straightforward for total hogs, pig crop and sows farrowed. The four market weight groups add complexity because they are included as nonlinear functions in the *state*. If we use the ASB estimates in the weight group functions as the measurements for the weight groups, we can eliminate the nonlinearities in the system equations.

For f_1, we have

$$\begin{gathered} G_{1_t}^{ASB} + G_{2_t}^{ASB} = G_{1_t} + G_{2_t} + v_t \\ f_{1_t}^{ASB} = f_{1_t} + v_t \end{gathered} \tag{29}$$

For f_2, we have

$$\ln\left(\frac{G_{1_t}^{ASB} + \alpha_t G_{2_t}^{ASB}}{P_t^{ASB} + C_t}\right) = \ln\left(\frac{G_{1_t} + \alpha_t G_{2_t}}{P_t + C_t}\right) + v_t$$
$$f_{2_t}^{ASB} = f_{2_t} + v_t \tag{30}$$

For f_3, we have

$$\ln\left[(1-\alpha_t)G_{2_t}^{ASB} + G_{3_t}^{ASB} + G_{4_t}^{ASB}\right] = \ln\left[(1-\alpha_t)G_{2_t} + G_{3_t} + G_{4_t}\right] + v_t$$
$$f_{3_t}^{ASB} = f_{3_t} + v_t \tag{31}$$

For f_4, we have

$$\ln\left(G_{4_t}^{ASB}\right) = \ln\left(G_{4_t}\right) + v_t$$
$$f_{4_t}^{ASB} = f_{4_t} + v_t \tag{32}$$

10.1.2 ASB Expert Analysis as a Biased Observation

An argument can be made that the ASB estimates are also biased. If this is the case, we can also treat the bias as an unobserved signal and measure it in the *state*. Inclusion of the bias in the *state* requires a transition model for the bias terms in the system of transition equations. This parameterization is important in the analysis of the decomposition and the influence of ASB measurements in the observation vector which will be shown in section 13. We will be able to compare the influence of the ASB measurements on the estimates in the case that the ASB is treated as unbiased against the case in which the ASB measurements are allowed to contain possible bias. We express biased ASB measurements as

$$x_t^{ASB} = x_t + b_{x_t} + v_t$$
$$b_{x_t} = b_{x_{t-1}} + w_t \tag{33}$$

where b_{x_t} is the bias term of item x and is transitioned as a random walk.

For the initial conditions of the filter, we set the bias parameters of all ASB measurements of inventory items to zero, allowing the filter to assess the biases starting with an initial assumption of zero bias.

10.1.3 Published Inventory as an Observation

Transitioning operationally from ASB published to Filter published is possible without the inclusion of ASB measurements of inventory when the published is treated as an unbiased measurement of inventory. The ASB sets an estimate for the current quarter and makes revisions to past estimates up to two quarters back in time. The backward revisions are due to the new slaughter data which give information on those hog inventories two quarters in the past. This can be implemented by the Filter operationally if the Kalman Filter estimates two quarters or more in the past become the observations. This is essentially the equivalent of using the fixed lag smoother $E[x_{t-2}|y_t]$ as a measurement in the observation vector. Given that $t = k$ is the quarter of the last published ASB estimates before implementation of the Kalman Filter and $t = n$ is the most recent estimate quarter, the published inventory observation is defined below in equation (34).

$$x_t^{PUB} = x_t + v_t$$

$$x_t^{PUB} = \begin{cases} x_t^{ASB} & 1 \leq t \leq k \\ E[x_t | y_{t+2}] & k < t < n-2 \\ missing & t \geq n-2 \end{cases} \tag{34}$$

For any time point equal to or more recent than two quarters in the past relative to the target quarter of estimation $t = n$, the published observation is coded as missing in the Kalman Filter. This is because a fixed lag smoother of lag 2 is not yet available for $t \in \{n-1, n\}$ and the observation will be created for $t = n-2$ during the quarter.

10.2 Survey Estimates and True Inventory

Figure 1-Figure 10 illustrate the degree of bias between the ASB estimate and the survey estimates for each inventory item. That bias demonstrated by the difference between survey results and the ASB published inventory estimates changes over time. In order to account for the bias in the survey measurements, we estimate it as part of the *state*. If x_t^{ADXX} represents the ADXX survey result for inventory item x, we write

$$x_t^{ADXX} = x_t + u_{x_t} + v_t \tag{35}$$

The above representation of the survey result for inventory item *x* shows the decomposition into the true inventory plus the bias term plus an observational noise process. This is a similar decomposition to the treatment of the biased ASB estimate in section 10.1.2 except that we use u to represent the survey bias and b to represent the ASB bias. As the bias is part of the *state*, we also need a transition model which defines how the bias is correlated over time (if at all). If we assume that the ASB published estimate for total hogs and pigs is true inventory, the bias for the ADXX list frame survey result for total hogs and pigs can be modeled with an ARIMA(1,0,0)x(0,1,1)$_4$. This bias model fit is demonstrated in Figure 32.

Figure 32

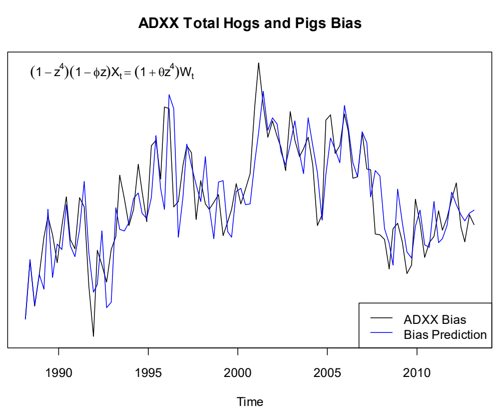
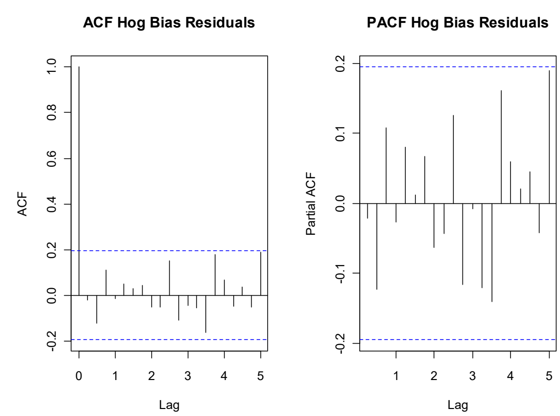


Figure 33



An ARIMA(0,1,0)x(0,1,1)$_4$ could slightly reduce Akaike's Information Criterion for the fit, however, does not fully remove autocorrelation in the residuals. Figure 33 gives the autocorrelation function and partial autocorrelation function of the residuals of the model fit from Figure 32, thereby providing evidence that this model removes significant autocorrelation structure.

Pig crop and sows farrowed ADXX biases could be modeled similarly conditioned on the assumption that the ASB is unbiased. This ARIMA model does add significant complexity to the state-space model, particularly in dramatically increasing the dimension of the *state* in order to reflect this transition over multiple inventory items. Additionally it would require estimation of the measurement matrix in order to reflect the seasonal moving average terms. It is more reasonable to assume that the ASB estimates are in fact estimates of true inventory and may possibly contain biases of their own. In light of this assumption, we would not want to use a transition model that reflected biased bias terms. For the sake of model parsimony, we represent the survey bias term transitions for all survey inventory items with a random walk. Equation (36) is the transition of the survey bias term.

$$u_{x_t} = u_{x_{t-1}} + v_t \tag{36}$$

For the survey litter rate, we base the observation equation on the ASB assumption

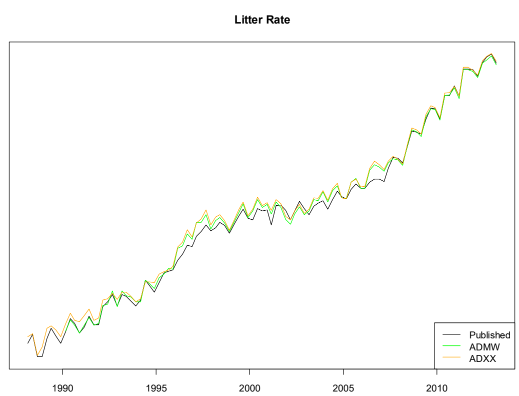
$$E\left[\frac{P_t^{ADXX}}{S_t^{ADXX}}\right] \approx \frac{P_t}{S_t}$$

This implies that the ASB believes the survey litter rate is unbiased. The observation equation for the litter rate with multiplicative observational noise becomes

$$\begin{gathered} \frac{P_t^{ADXX}}{S_t^{ADXX}} = \frac{P_t}{S_t} e^{v_t} \\ \ln(T_t^{ADXX}) = \ln(P_t) - \ln(S_t) + v_t \end{gathered} \tag{37}$$

The relationship between the ASB litter rate and the survey litter rates is shown in Figure 34.

Figure 34



10.3 External Data and True Inventory

Section 4 listed all of the hog inventory constraints. These constraints define how inventories relate to the various inventory items, and also how they relate to a number of external data items. The death loss ratio constraint from 4.2 and the weight group transition constraint from 4.3 were included in the transition equation. These constraints involved transitional relationships of the multidimensional *state* over time. The remainder of the constraints from section 4 involves external transaction data and can be expressed in the observation equation.

10.3.1 Balance Sheet Equation

The balance sheet constraint was introduced in section 4.1 and the balance sheet net was defined in Table 2 of section 3. The balance sheet relationship is

$$H_t = H_{t-1} + P_t + BSN_t + BSR_t \tag{38}$$

If we define $BSR_t = -v_t$, the above can be written as

$$BSN_t = H_t - H_{t-1} - P_t + v_t \tag{39}$$

The balance sheet equation is now in state-space form with the observation noise process equated to the balance sheet residual. In terms of elements of the state vector, the three month balance sheet measurement equation is

$$BSN_t = \Delta(H_t) + H_{t-4} - H_{t-5} - \Delta(P_t) - P_{t-1} - P_{t-4} + P_{t-5} + v_t \tag{40}$$

The six month balance sheet measurement equation is

$$\sum_{k=0}^{1} BSN_{t-k} = \Delta(H_t) + H_{t-1} + H_{t-4} - H_{t-5} - H_{t-2} - \Delta(P_t) - 2P_{t-1} - P_{t-4} + P_{t-5} + v_t \tag{41}$$

The twelve month balance sheet measurement equation is

$$\sum_{k=0}^{3} BSN_{t-k} = \Delta(H_t) + H_{t-1} - H_{t-5} - \Delta(P_t) - 2P_{t-1} - P_{t-2} - P_{t-3} - P_{t-4} + P_{t-5} + v_t \tag{42}$$

10.3.2 Breeding Herd "Smoother"

Stricter constraints on the market weight groups and total hogs and pigs cause breeding herd to absorb some of the noise process due to the deterministic relationship they share. When uncontrolled, it is manifested in high frequency oscillations. Contrary to what these high frequency oscillations suggest, breeding herd should be a more stable inventory item, meaning that it does not change dramatically from quarter to quarter. This can be represented as an observation constraint where the change in breeding herd has "fixed" bounds. The measurement for this constraint is 1, and the bounds can be set by a fixed diagonal element of the covariance matrix $R = E[v_t v_t']$. The constraint is then

$$\begin{aligned} 1 &= \frac{B_t}{B_{t-1}} e^{v_t} \\ 0 &= \ln(B_t) - \ln(B_{t-1}) + v_t \end{aligned} \tag{43}$$

The remainder of the external constraints was introduced in section 4. Table **5** lists all observation equations as a function of the *state* specified in Table **3**. The observation noise term is omitted for convenience.

Table 5

Item	Section	Obs Vector Element	Observation Equation	
1	0	X_t^{ASB}	$X_t^{ASB} = \Delta(X_t) + X_{t-1} + X_{t-4} - X_{t-5}$	$X \in \{H, P, S\}$
2	0	$f_{1_t}^{ASB}$	$f_{1_t}^{ASB} = \Delta(f_{1_t}) + f_{1_{t-1}} + f_{1_{t-4}} - f_{1_{t-5}}$	
3	0	$f_{2_t}^{ASB}$	$f_{2_t}^{ASB} = f_{2_t}$	
4	0	$f_{3_t}^{ASB}$	$f_{3_t}^{ASB} = f_{3_t}$	
5	0	$f_{4_t}^{ASB}$	$f_{4_t}^{ASB} = f_{4_t}$	
6	10.2	X_t^{ADMW}	$X_t^{ADMW} = X_t + u_{X_t}^{ADMW}$	$X \in \{H, P, S\}$
7	10.2	X_t^{ADXX}	$X_t^{ADXX} = X_t + u_{X_t}^{ADXX}$	$X \in \{H, P, S\}$
8	10.2	$\ln\left(\frac{P_t^X}{S_t^X}\right)$	$\ln\left(\frac{P_t^X}{S_t^X}\right) = \ln(P_t) - \ln(S_t)$	$X \in \{ADMW, ADXX\}$
10	0	BSN_t	$BSN_t = \Delta(H_t) + H_{t-4} - H_{t-5} - \Delta(P_t) - P_{t-1} - P_{t-4} + P_{t-5}$	
11	0	$\sum_{k=0}^{1} BSN_{t-k}$	$\sum_{k=0}^{1} BSN_{t-k} = \Delta(H_t) + H_{t-1} + H_{t-4} - H_{t-5} - H_{t-2} - \Delta(P_t) - 2P_{t-1} - P_{t-4} + P_{t-5}$	
12	0	$\sum_{k=0}^{3} BSN_{t-k}$	$\sum_{k=0}^{3} BSN_{t-k} = \Delta(H_t) + H_{t-1} - H_{t-5} - \Delta(P_t) - 2P_{t-1} - P_{t-2} - P_{t-3} - P_{t-4} + P_{t-5}$	
13	4.4	$\ln\left(\frac{L_t}{L_{t-4}}\right)$	$\ln\left(\frac{L_t}{L_{t-4}}\right) = \ln\left(\frac{P_{t-2}}{\Delta(P_{t-1}) - P_{t-1} + P_{t-2} + P_{t-5}}\right)$	
14	4.5	$\ln\left(\frac{L_t + L_{t-1}}{L_{t-4} + L_{t-5}}\right)$	See Market Slaughter*	
15	4.6	$\ln\left(\frac{L_{t+1}^{5WK}}{L_{t-3}^{5WK}}\right)$	$\ln\left(\frac{L_{t+1}^{5WK}}{L_{t-3}^{5WK}}\right) = f_{4_t} - f_{4_{t-4}}$	
16	4.7	$\ln(.5)$	$\ln(.5) = \ln\left\{\frac{\Delta(S_t) + S_{t-1} + S_{t-4} - S_{t-5}}{H_{t-1} - \left[(P_{t-1} + C_{t-1})e^{f_{2_{t-1}}} + e^{f_{3_{t-1}}}\right]}\right\}$	
17	10.3.2	0	See Breeding Herd Smoother**	

*Market Slaughter

$$\ln\left(\frac{L_t + L_{t-1}}{L_{t-4} + L_{t-5}}\right) = \ln\left[\frac{(P_{t-2} + C_{t-2})e^{f_{2_{t-2}}} + e^{f_{3_{t-2}}}}{[\Delta(P_{t-1}) - P_{t-1} + P_{t-2} + P_{t-5} + C_{t-6}]e^{f_{2_{t-6}}} + e^{f_{3_{t-6}}}}\right]$$

**Breeding Herd Smoother

$$0 = \ln\left\{\frac{\Delta(H_t) + H_{t-1} + H_{t-4} - H_{t-5} - \left[(\Delta(P_t) + P_{t-1} + P_{t-4} - P_{t-5} + C_t)e^{f_{2_t}} + e^{f_{3_t}}\right]}{H_{t-1} - \left[(P_{t-1} + C_{t-1})e^{f_{2_{t-1}}} + e^{f_{3_{t-1}}}\right]}\right\}$$

11 Kalman Filter Estimates of U.S. Level Hog Inventories

Given the hog inventory system equations in State-Space form from sections 9 and 10, the Extended Kalman Filter is used to estimate the state *vector and its standard errors. The* state *vector contains functions of inventory items, and the standard errors estimated by the filter are standard errors of the functions of inventory items. This section covers the transformations from* state *vector to U.S.-level inventory items.*

If $t = n$ represents the most recent target survey period of reference, we are interested in estimating the vector

$$q_n = [H_n \quad P_n \quad S_n \quad G_{1_n} \quad G_{2_n} \quad G_{3_n} \quad G_{4_n} \quad M_n \quad B_n \quad T_n]'$$

and its variance. The vector q_n represents all published inventory items from Table 1 at the U.S. level. The vector of *state* elements at time $t = n$ from Table **3** is however

$$x_n = \left[\Delta(H_n) \quad \Delta(P_n) \quad \Delta(S_n) \quad \Delta\left(f_{1_n}\right) \quad f_{2_n} \quad f_{3_n} \quad f_{4_n} \dots\right]'$$

We express the vector of inventory items to be published as a vector of functions of the *state* vector.

$$q_n = F(x_n)$$

The vector of functions $F(x_n)$ contains both linear and nonlinear functions. The Extended Kalman Filter (EKF) uses a first order Taylor Series approximation of the first and second central moments. For linear functions, this will be the exact mean and variance. A first order Taylor Series expansion of hog inventories $q_n = F(x_n)$ about $E[x_n]$ is

$$F(x_n) = F(E[x_n]) + F^{\nabla}(E[x_n])(x_n - E[x_n])$$

where $F^{\nabla}(\cdot)$ represents the Jacobian of $F(\cdot)$. Taking the first order approximation and subtracting its expectation yields

$$F(x_n) - E[F(x_n)] = F^{\nabla}(E[x_n])(x_n - E[x_n])$$
$$var[F(x_n)] = \Sigma_{q_n} = F^{\nabla}(E[x_n])P_n F^{\nabla}(E[x_n])'$$

The first order Taylor Series approximation of q_n and its covariance matrix are therefore

$$q_n = F(E[x_n]) \tag{44}$$
$$\Sigma_{q_n} = J_n P_n J_n' \tag{45}$$

where $J_n = F^{\nabla}(E[x_n])$, and $P_n = E[(x_n - E[x_n])(x_n - E[x_n])']$.

The vector of functions of Kalman Filter estimates of hog inventories $q_t = F(x_t)$ is as follows:

12 State-Level Inventories

This section covers Restricted Least Squares allocation of the Kalman Filter estimates of U.S.-level inventories to the states. State-level survey results must be calibrated to sum to the U.S. estimates for each of the inventory items with the exception of the litter rate.

At the state level, there is no formal ASB panel analogous to the one at the U.S. level. However, the state commodity analysts set a “state recommendation” which most often differs from the survey results. When a state believes its survey results to be biased, they choose to recommend inventory estimates differing from the survey results. This is often due to significant undercoverage or nonresponse from extreme operators – those farms which are so large, they form a majority of the hog production in a given state. The constraints that we have formulated in State-Space representation for the Kalman Filter estimates of U.S.-level hog inventories are U.S.-level constraints. This is attributed to the fact that the external data is available at the U.S. level only. The states’ inventory estimates must sum to the U.S.-level inventory estimates for total hogs, pig crop, sows farrowed, the market weight groups, and breeding herd. The state-level estimates must be adjusted for this constraint to hold. This is accomplished using Restricted Least Squares techniques.

We now establish notations and definitions in order to derive the Restricted Least Squares methodology for estimating state-level inventory. The time subscript *t* may be omitted for convenience unless $t \neq n$.

Notation	Definition
$x_{n\|n}$	The expected value of the *state* at time $t = n$ given measurements of the state at time $t \in \{1,2,\dots,n\}$ or $E[x_n\|y_n]$. This is the vector of Kalman Filter estimates of the *state* for the most recent estimation quarter for hog inventory items.
$P_{n\|n}$	The covariance matrix of $x_{n\|n}$ from the Kalman Filter. $P_{n\|n} = E\left[(x_n - x_{n\|n})(x_n - x_{n\|n})'\|y_n\right]$
y_n	Vector of measurements of inventory for time $t = n$.
β_{SV}	Vector of list frame survey results ordered by inventory item and state at time $t = n$.
β_{SV}^*	Vector of list frame survey results ordered by subset of inventory items and state at time $t = n$. The subset of inventory items consists of pig crop (P), sows farrowed (S), market hogs <50 lbs (G1), market hogs 50-119 lbs (G2), market hogs 120-179 lbs (G4), market hogs over 180 lbs (G4), and breeding herd (B).
$\overline{\beta_{SV}^*}$	Vector of list frame survey results ordered by subset of inventory items and state at time $t = n$. The subset of inventory items consists of total hogs and pigs (H), total market hogs (M), and litter rate (T). These are the inventory items that can be derived from β_{SV}^*.
β_{SR}^*	Vector of state recommendations ordered by subset of inventory item and state at time $t = n$. The subset of inventory items consists of pig crop (P), sows farrowed (S), market hogs <50 lbs (G1), market hogs 50-119 lbs (G2), market hogs 120-179 lbs (G3), market hogs over 180 lbs (G4), and breeding herd (B).
β_{KF}	Vector of state-level estimates of inventory derived from the U.S.-level Kalman Filter estimates of inventory using Restricted Least Squares allocation. The vector elements are ordered by inventory item and state.
β_{KF}^*	Vector of state-level estimates of a subset of inventory items derived from the U.S.-level Kalman Filter estimates of inventory using Restricted Least Squares allocation. The vector elements are ordered by inventory item and state. The subset of inventory items consists of pig crop (P), sows farrowed (S), market hogs <50 lbs (G1), market hogs 50-119 lbs (G2), market hogs 120-179 lbs (G4), market hogs over 180 lbs (G4), and breeding herd (B).
$\overline{\beta_{KF}^*}$	Vector of state-level estimates of a subset of inventory items derived from the U.S.-level Kalman Filter estimates of inventory using Restricted Least Squares allocation. The vector elements are ordered by inventory item and state. The subset of inventory items consists of total hogs and pigs (H), total market hogs (M), and litter rate (T).

Notation	Definition
$\delta(\beta^*_{KF})$	The vector of functions that calculates $\overline{\beta^*_{KF}}$ from β^*_{KF}. $$\delta(\beta^*_{KF}) = \begin{bmatrix} B_t + G_{1_t} + G_{2_t} + G_{3_t} + G_{4_t} \\ G_{1_t} + G_{2_t} + G_{3_t} + G_{4_t} \\ \frac{P_t}{S_t} \end{bmatrix} = \begin{bmatrix} H_t \\ M_t \\ T_t \end{bmatrix}$$
$\delta^{\nabla}(\beta^*_{KF})$	The Jacobian of $\delta(\beta^*_{KF})$.
D	Diagonal weight matrix containing the difference $\beta^*_{SR} - \beta^*_{SV}$
q_{SV}	Vector of list frame survey results at the U.S. level at time $t = n$.
q^*_{SV}	Vector of a subset of list frame survey results at the U.S. level at time $t = n$. The subset of inventory items consists of pig crop (P), sows farrowed (S), market hogs <50 lbs (G1), market hogs 50-119 lbs (G2), market hogs 120-179 lbs (G4), market hogs over 180 lbs (G4), and breeding herd (B).
$\overline{q^*_{SV}}$	Vector of a subset of list frame survey results at the U.S. level at time $t = n$. The subset of inventory items consists of total hogs and pigs (H), total market hogs (M), and litter rate (T).
q_{KF}	Vector of Kalman Filter estimates of hog inventory at the U.S. level at time $t = n$. $q_{KF} = F(x_{n\|n})$.
q^*_{KF}	Vector of a subset of Kalman Filter estimates of hog inventory at the U.S. level at time $t = n$. The subset of inventory items consists of pig crop (P), sows farrowed (S), market hogs <50 lbs (G1), market hogs 50-119 lbs (G2), market hogs 120-179 lbs (G4), market hogs over 180 lbs (G4), and breeding herd (B). $q^*_{KF} = F^*(x_{n\|n})$.
$\overline{q^*_{KF}}$	Vector of a subset of Kalman Filter estimates of hog inventory at the U.S. level at time $t = n$. The subset of inventory items consists of total hogs and pigs (H), total market hogs (M), and litter rate (T). $\overline{q^*_{KF}} = \overline{F^*}(x_{n\|n})$
U	A linear operator such that $q^*_{SV} = U\beta^*_{SV}$ and $q^*_{KF} = U\beta^*_{KF}$. $U = [\boldsymbol{I}\otimes\boldsymbol{1}]'$ where $\boldsymbol{1}$ is a 50×1 vector of 1s and $\boldsymbol{I}$ is a 7×7 identity matrix representing the seven inventory items $P, S, G_1, G_2, G_3, G_4, B$.
Σ_x	The covariance matrix of x, or $E[(x - E[x])(x - E[x])']$
Σ_{xy}	The covariance matrix of x and y, or $E[(x - E[x])(y - E[y])']$
K_n	The Kalman Gain from the Kalman Filter at time $t = n$.
J_n	The Jacobian of the vector of functions $F(x_{n\|n})$.
J^*_n	The Jacobian of the vector of functions $F^*(x_{n\|n})$
p_j	Pig crop for state j.
s_j	Sows farrowed for state j.
$\sum_j p$	Pig crop at the U.S. level summed over all states.
$\sum_j s$	Sows farrowed at the U.S. level summed over all states.
$\sigma^2_{p_j}$	Variance of list frame survey result for pig crop for state j.
$\sigma^2_{s_j}$	Variance of list frame survey result for sows farrowed for state j.
$\sigma_{p_j s_j}$	Covariance of list frame survey results for pig crop and sows farrowed for state j. $\sigma_{p_j s_j} = \sigma_{s_j p_j}$

We define β_{ij} to be the state-level inventory for inventory item i from Table 1 state j. $\beta = [\beta_{ij}]'$ for $i \in \{1,2,\ldots,10\}$ and $j \in \{1,2,\ldots,50\}$ ordered by i then j. $\beta^* = [\beta_{ij}]'$ for $i \in [2,3,4,5,6,7,9]$. $\overline{\beta^*}$ will be used to denote the complement of β^*. $\beta_{SR} = \beta^*_{SR} \cup \overline{\beta^*_{SR}}$ represents the vector of state recommendations, and $\beta_{SV} = \beta^*_{SV} \cup \overline{\beta^*_{SV}}$ represents the vector of state survey results. We define $U = [I\otimes 1]'$ where 1 is a 50×1 vector of 1s and I is a 7×7 identity matrix. $q^*_{SR} = U\beta^*_{SR}$ is the U.S.-level summed state recommendation vector for the appropriate subset of inventory items and $q^*_{SV} = U\beta^*_{SV}$ is the U.S.-level summed survey results vector of the subset of inventory items. The vector q_{KF} contains all hog inventory items estimated by the Kalman Filter at the U.S. level. The Restricted Least Squares

(RLS) estimate for the Ordinary Least Squares (OLS) regression parameter in the model $y = X\beta + E$ with $E \sim N(0, \Sigma_E)$ is

$$\beta_{RLS} = \beta_{OLS} + (X'X)^{-1}U'[U(X'X)^{-1}U']^{-1}(q - U\beta_{OLS}) \tag{46}$$

A derivation can be found in Green (2000). Equation (46) can be rewritten as

$$\beta_{KF}^* = \beta_{SV}^* + DU'[UDU']^{-1}(q_{KF}^* - U\beta_{SV}^*) \tag{47}$$

where $D = diag(\beta_{SR}^* - \beta_{SV}^*)$ is a diagonal weight matrix. This formulation creates the adjustments to the survey results according to the assumed degree of bias. If the state recommendations are the state survey results, the diagonal element or weight is equal to zero and the RLS adjustment is zero. The survey results are published. The degree of adjustment to the survey results is therefore correlated with the assumed bias. The Kalman Filter Restricted Least Squares state-level hog inventory estimates are

$$\begin{aligned} \beta_{KF}^* &= (I - ZU)\beta_{SV}^* + Zq_{KF}^* \\ \overline{\beta_{KF}^*} &= \delta(\beta_{KF}^*) \\ \beta_{KF} &= [\beta_{KF}^* \quad \overline{\beta_{KF}^*}]' \end{aligned}$$

where $Z = DU'[UDU']^{-1}$ and $\delta(\cdot)$ is the vector of functions that gives total hogs (summation), total market hogs (summation), and litter rate (ratio of sums). The covariance matrix is given by

$$\begin{aligned} \Sigma_{\beta_{KF}^*} &= (I - ZU)\Sigma_{\beta_{SV}^*}(I - ZU)' + ZJ_n^*P_nJ_n^{*\prime}Z' + (I - ZU)\Sigma_{\beta_{SV}^*Y_n}K_n'J_n^{*\prime}Z' + ZJ_n^*K_n\Sigma_{Y_n\beta_{SV}^*}(I - ZU)' \\ \Sigma_{\overline{\beta_{KF}^*}} &= \delta^\nabla(\beta_{KF}^*)\Sigma_{\beta_{KF}^*}\delta^\nabla(\beta_{KF}^*)' \\ \Sigma_{\beta_{KF}^*\overline{\beta_{KF}^*}} &= (I - ZU)\left[\Sigma_{\beta_{SV}^*}(I - ZU)' + \Sigma_{\beta_{SV}^*Y_n}K_n'J_n^{*\prime}Z'\right]\delta^\nabla(\beta_{KF}^*)' + Z\left[J_n^*K_n\Sigma_{Y_n\beta_{SV}^*}(I - ZU)' + J_n^*P_nJ_n^{*\prime}Z'\right]\delta^\nabla(\beta_{KF}^*)' \\ \Sigma_{\beta_{KF}} &= \begin{bmatrix} \Sigma_{\beta_{KF}^*} & \Sigma_{\beta_{KF}^*\overline{\beta_{KF}^*}} \\ \Sigma_{\overline{\beta_{KF}^*}\beta_{KF}^*} & \Sigma_{\overline{\beta_{KF}^*}} \end{bmatrix} \end{aligned}$$

In order to derive the covariance matrix of the state allocation, we first need to derive the covariance of the survey results with the estimated state $\Sigma_{\beta_{SV}^*x_{n|n}} = cov(\beta_{SV}^*, x_{n|n})$.

Derivation of $\Sigma_{\beta_{SV}^*x_{n|n}}$

$$\begin{aligned} \Sigma_{\beta_{SV}x_{n|n}} &= E\left[(\beta_{SV}^* - E[\beta_{SV}^*])\left(x_{n|n} - E\left[x_{n|n}\right]\right)'\right] \\ x_{n|n} &= x_{n|n-1} - K_ng\left(x_{n|n-1}\right) + K_nY_n \\ E\left[x_{n|n}\right] &= E\left[x_{n|n-1} - K_ng\left(x_{n|n-1}\right)\right] + K_nE[Y_n] \\ x_{n|n} - E\left[x_{n|n}\right] &= x_{n|n-1} - K_ng\left(x_{n|n-1}\right) - E\left[x_{n|n-1} - K_ng\left(x_{n|n-1}\right)\right] + K_n(Y_n - E[Y_n]) \\ \Sigma_{\beta_{SV}^*x_{n|n}} &= \Sigma_{\beta_{SV}^*\gamma(x_{n|n-1})} + \Sigma_{\beta_{SV}^*Y_n}K_n' \\ &= \Sigma_{\beta_{SV}^*Y_n}K_n' \end{aligned}$$

The final result is because $\Sigma_{\beta_{SV}\gamma(x_{n|n-1})} = 0$ as the survey results at time *n* are independent and therefore not correlated with the prediction of the *state* at time *n;* nor any function thereof. All terms with $x_{n|n-1}$ are collapsed into the function $\gamma\left(x_{n|n-1}\right)$. The covariance matrix $\Sigma_{\beta_{SV}Y_n}$ is nonzero for the U.S. survey result observations (sum and litter rate). Otherwise it is zero.

Derivation of $\Sigma_{\ {}^{*}_{SV} q^{*}_{KF}}$

$$
\begin{aligned}
\Sigma_{\ {}^{*}_{SV} q^{*}_{KF}} &= E[(\beta^{*}_{SV} - E[\beta^{*}_{SV}])(q^{*}_{KF} - E[q^{*}_{KF}])'] \\
q^{*}_{KF} &= F^{*}\left(x_{n|n}\right) \\
q^{*}_{KF} &= F^{*}\left(E\left[x_{n|n}\right]\right) + F^{*\nabla}\left(E\left[x_{n|n}\right]\right)\left(x_{n|n} - E\left[x_{n|n}\right]\right) \\
E[q^{*}_{KF}] &= F^{*}\left(E\left[x_{n|n}\right]\right) \\
q^{*}_{KF} - E[q^{*}_{KF}] &= F^{*\nabla}\left(E\left[x_{n|n}\right]\right)\left(x_{n|n} - E\left[x_{n|n}\right]\right) \\
\Sigma_{\ {}^{*}_{SV} q^{*}_{KF}} &= \Sigma_{\ {}^{*}_{SV} x_{n|n}} F^{*\nabla}\left(x_{n|n}\right)' \\
&= \Sigma_{\ {}^{*}_{SV} Y_n} K_n' J_n^{*\prime}
\end{aligned}
$$

Derivation of $\Sigma_{\ {}^{*}_{KF}}$

$$
\begin{aligned}
\Sigma_{\ {}^{*}_{KF}} &= E[(\beta^{*}_{KF} - E[\beta^{*}_{KF}])(\beta^{*}_{KF} - E[\beta^{*}_{KF}])'] \\
\beta^{*}_{KF} &= (I - ZU)\beta^{*}_{SV} + Zq^{*}_{KF} \\
E[\beta^{*}_{KF}] &= (I - ZU)E[\beta^{*}_{SV}] + ZE[q^{*}_{KF}] \\
\beta^{*}_{KF} - E[\beta^{*}_{KF}] &= (I - ZU)(\beta^{*}_{SV} - E[\beta^{*}_{SV}]) + Z(q^{*}_{KF} - E[q^{*}_{KF}]) \\
\Sigma_{\ {}^{*}_{KF}} &= (I - ZU)\Sigma_{\ {}^{*}_{SV}}(I - ZU)' + Z\Sigma_{q^{*}_{KF}}Z' + (I - ZU)\Sigma_{\ {}^{*}_{SV} q^{*}_{KF}}Z' + Z\Sigma_{\ {}^{*}_{SV} q^{*}_{KF}}(I - ZU)' \\
&= (I - ZU)\Sigma_{\ {}^{*}_{SV}}(I - ZU)' + ZJ_n^{*}P_nJ_n^{*\prime}Z' + (I - ZU)\Sigma_{\ {}^{*}_{SV} Y_n}K_n'J_n^{*\prime}Z' + ZJ_n^{*}K_n\Sigma_{Y_n\ {}^{*}_{SV}}(I - ZU)'
\end{aligned}
$$

Derivation of $\Sigma_{\ {}^{*}_{SV}\ {}^{*}_{KF}}$

$$
\begin{aligned}
\Sigma_{\ {}^{*}_{SV}\ {}^{*}_{KF}} &= E[(\beta^{*}_{SV} - E[\beta^{*}_{SV}])(\beta^{*}_{KF} - E[\beta^{*}_{KF}])'] \\
(\beta^{*}_{KF} - E[\beta^{*}_{KF}])' &= (\beta^{*}_{SV} - E[\beta^{*}_{SV}])'(I - ZU)' + (q^{*}_{KF} - E[q^{*}_{KF}])'Z' \\
\Sigma_{\ {}^{*}_{SV}\ {}^{*}_{KF}} &= (\beta^{*}_{SV} - E[\beta^{*}_{SV}])(\beta^{*}_{SV} - E[\beta^{*}_{SV}])'(I - ZU)' + (\beta^{*}_{SV} - E[\beta^{*}_{SV}])(q^{*}_{KF} - E[q^{*}_{KF}])'Z' \\
&= \Sigma_{\ {}^{*}_{SV}}(I - ZU)' + \Sigma_{\ {}^{*}_{SV} q^{*}_{KF}}Z' \\
&= \Sigma_{\ {}^{*}_{SV}}(I - ZU)' + \Sigma_{\ {}^{*}_{SV} Y_n}K_n'J_n^{*\prime}Z'
\end{aligned}
$$

Derivation of $\Sigma_{q^{*}_{KF}\ {}^{*}_{KF}}$

$$
\begin{aligned}
q^{*}_{KF} - E[q^{*}_{KF}] &= J_n^{*}\left(x_{n|n} - E\left[x_{n|n}\right]\right) \\
(\beta^{*}_{KF} - E[\beta^{*}_{KF}])' &= (\beta^{*}_{SV} - E[\beta^{*}_{SV}])'(I - ZU)' + (q^{*}_{KF} - E[q^{*}_{KF}])'Z' \\
\Sigma_{q^{*}_{KF}\ {}^{*}_{KF}} &= J_n^{*}\Sigma_{x_{n|n}\ {}^{*}_{SV}}(I - ZU)' + J_n^{*}\Sigma_{x_{n|n}q^{*}_{KF}}Z' \\
&= J_n^{*}K_n\Sigma_{Y_n\ {}^{*}_{SV}}(I - ZU)' + J_n^{*}P_nJ_n^{*\prime}Z'
\end{aligned}
$$

Derivation of $\Sigma_{\ {}^{*}_{KF}\ \overline{{}^{*}_{KF}}}$

$$
\begin{aligned}
\Sigma_{\ {}^{*}_{KF}\ \overline{{}^{*}_{KF}}} &= E\left[(\beta^{*}_{KF} - E[\beta^{*}_{KF}])\left(\overline{\beta^{*}_{KF}} - E\left[\overline{\beta^{*}_{KF}}\right]\right)'\right] \\
\beta^{*}_{KF} - E[\beta^{*}_{KF}] &= (I - ZU)(\beta^{*}_{SV} - E[\beta^{*}_{SV}]) + Z(q^{*}_{KF} - E[q^{*}_{KF}]) \\
\left(\overline{\beta^{*}_{KF}} - E\left[\overline{\beta^{*}_{KF}}\right]\right)' &= (\beta^{*}_{KF} - E[\beta^{*}_{KF}])'\delta^{\nabla}(\beta^{*}_{KF})' \\
\Sigma_{\ {}^{*}_{KF}\ \overline{{}^{*}_{KF}}} &= (I - ZU)\Sigma_{\ {}^{*}_{SV}\ {}^{*}_{KF}}\delta^{\nabla}(\beta^{*}_{KF})' + Z\Sigma_{q^{*}_{KF}\ {}^{*}_{KF}}\delta^{\nabla}(\beta^{*}_{KF})'
\end{aligned}
$$

$$F^{\nabla}(\mathrm{E[X]})\Sigma_{\mathrm{X}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{1}{\sum_j p} & \frac{1}{\sum_j p} & -\frac{1}{\sum_j s} & -\frac{1}{\sum_j s} \end{bmatrix} \begin{bmatrix} \sigma_{p_1}^2 & 0 & \sigma_{p_1 s_1} & 0 \\ 0 & \sigma_{p_2}^2 & 0 & \sigma_{p_2 s_2} \\ \sigma_{s_1 p_1} & 0 & \sigma_{s_1}^2 & 0 \\ 0 & \sigma_{s_2 p_2} & 0 & \sigma_{s_2}^2 \end{bmatrix} =$$

$$\begin{bmatrix} \sigma_{p_1}^2 & \sigma_{p_2}^2 & \sigma_{p_1 s_1} & \sigma_{p_2 s_2} \\ \sigma_{s_1 p_1} & \sigma_{s_2 p_2} & \sigma_{s_1}^2 & \sigma_{s_2}^2 \\ \frac{1}{\sum_j p}\sigma_{p_1}^2 - \frac{1}{\sum_j s}\sigma_{s_1 p_1} & \frac{1}{\sum_j p}\sigma_{p_2}^2 - \frac{1}{\sum_j s}\sigma_{s_2 p_2} & \frac{1}{\sum_j p}\sigma_{p_1 s_1} - \frac{1}{\sum_j s}\sigma_{s_1}^2 & \frac{1}{\sum_j p}\sigma_{p_2 s_2} - \frac{1}{\sum_j s}\sigma_{s_2}^2 \end{bmatrix}$$

These results can be expanded to any number of states.

13 Filter Results

This section provides results for three scenarios of Kalman Filter results. The results are from three different treatments or parameterizations of the ASB estimates in the Filter. For the first, we omit an ASB measurement. For the second treatment, we assume the ASB is biased. For the third treatment, we assume the ASB is unbiased. Details are provided on specific parameterizations of the observations so that it is clear how the treatments differ. Decomposition of the estimates into relative absolute net contributions is explained.

In this paper, we have defined the hog inventory system equations. Their derivations have been presented without any evidence of the performance of Filter inventory estimates by the various parameterizations. Performance can be evaluated more easily when there exist some unbiased measures of the true signal to which we can compare the filter results. Signal Filtering is used in situations where one or more measurements of an unobservable signal are collected, and by the very nature of the problem, the true signal is not available, so a comparison to truth cannot be made. This is the case with hog inventory estimation. As true inventories are not available for comparison, we will compare the Filter estimates to the ASB published, and to each other across treatments; and examine the inventory constraints to judge whether the Filter estimates follow the rules and "make sense". For each of the ten published inventory items; namely total hogs and pigs (H), pig crop (P), sows farrowed (S), market hogs less than 50 lbs (G_1), market hogs between 50 and 119 lbs (G_2), market hogs between 120 and 179 lbs (G_3), market hogs greater than 180 lbs (G_4), total market hogs (M), breeding herd (B), and litter rate (T); we will present graphically the results of a fixed interval Kalman Smoother estimate of inventory for three different parameterizations or treatments of the observations. We will compare the estimates of the variances of the Kalman Smoother estimates between the three treatments. We will also decompose the Smoother at the last point in time $E[x_t|y_n]|t = n$ into what will be defined as absolute relative net contributions from categories of observations. This will provide understanding as to which data items are influential in each treatment of the measurements. Additionally, we will include the estimates of the variances of the process noise and observation noise obtained by the Expectation Maximization algorithm for each of the three treatments. These are the diagonals of the Q and R matrices from sections 8.1 and sheet residuals are bounded by ± 500 thousand hogs. It is also essential to note that the ASB gives heavy weight to the survey litter rate. Omitting an ASB measurement and estimating the State-Space model parameters without a fixed constraint on the litter rate results in an observation noise estimate for the survey litter rate that is not consistent with ASB behavior. ASB behavior can be reflected in the Filter without ASB measurements by fixing the variance parameter associated with the survey litter rate observation equations. Fixing the variance in this way puts strict bounds on how far the filter estimate for the litter rate will deviate from the survey litter rate. For example, if the variance of the observation noise associated with the survey litter rate were fixed at zero, the filter estimate for the litter rate would be exactly the survey litter rate, and the filter estimates for pig crop and sows farrowed would reflect this. The farther that variance is fixed from zero, the more the filter estimate for litter rate can deviate from the survey litter rate. The breeding herd smoother is an imposed constraint and therefore also fixed. We treat the ratio between survey sows farrowed and previous quarter's breeding herd as unbiased. Although the sows farrowed/breeding herd constraint introduced in section 4.7 equation (9) defines sows farrowed as one half of the previous quarter's breeding herd, the

Figure **25** graph demonstrates that the historical ASB estimates do not agree. As the ASB treats the survey ratio of pig crop to sows farrowed as unbiased, we assume the survey ratio of sows farrowed to previous quarter's breeding herd is also unbiased. It is understood that the breeding herd in the denominator is from a different survey quarter and hence survey result than the numerator since it is lagged one quarter; however, the survey results are similar for the same survey quarter as for the lagged quarter as shown in Figure 35.

Figure 35

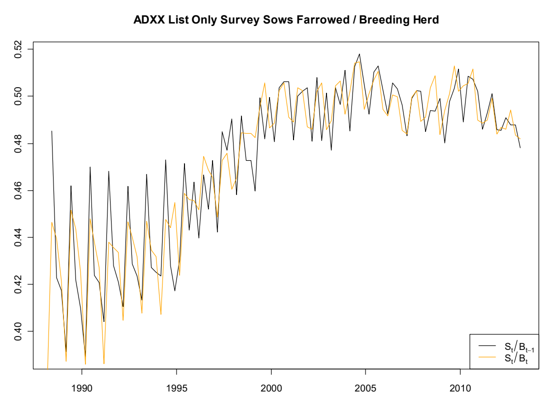


Table 7 below lists the hog inventory measurement notation, description, and how it is parameterized in the treatment. For these three treatment scenarios, we use only the ADXX list frame survey result with the exception of the litter rate. The ADMW multiframe survey result is not computed quarterly for all inventory items and it is sufficient to use the list frame survey result in the treatment scenarios.

Table 7

Notation	Description	Parameterization Of Observation Noise
ADXX.F4	List frame survey result for f_4	Estimated
BSN3	3 Month Balance Sheet Net	Fixed
BSN6	6 Month Balance Sheet Net	Fixed
BSN12	12 Month Balance Sheet Net	Fixed
RATIO.P	Slaughter Ratio – Pig Crop (section 4.4)	Estimated
RATIO.M	Slaughter Ratio – Market Hogs (section 4.5)	Estimated
SL5WKS	5 Week Slaughter Ratio (section 4.6)	Estimated
ADMW.SL	Multiframe Survey result for litter rate	Fixed
ADXX.SL	List frame survey result for litter rate	Fixed
.5	List frame survey result for S_t/B_{t-1}	Fixed
1	Breeding Herd Smoother (section 10.3.2)	Fixed

13.2 Treatment 2 – ASB Measurements as Biased Estimates

For the second scenario, Treatment 2, we add the ASB measurements to the observations and use the parameterization introduced in 10.1.2 which defines the ASB measurements as biased. The other items remain the same.

Table 8

Notation	Description	Parameterization Of Observation Noise
HP	ASB estimate for total hogs and pigs	Estimated
PP	ASB estimate for pig crop	Estimated
SP	ASB estimate for sows farrowed	Estimated
F1.P	ASB estimate for f_1	Estimated
F2.P	ASB estimate for f_2	Estimated
F3.P	ASB estimate for f_3	Estimated
F4.P	ASB estimate for f_4	Estimated
ADXX.H	List frame survey result for total hogs and pigs	Estimated
ADXX.P	List frame survey result for pig crop	Estimated
ADXX.S	List frame survey result for sows farrowed	Estimated
ADXX.F1	List frame survey result for f_1	Estimated
ADXX.F2	List frame survey result for f_2	Estimated
ADXX.F3	List frame survey result for f_3	Estimated
ADXX.F4	List frame survey result for f_4	Estimated
BSN3	3 Month Balance Sheet Net	Fixed
BSN6	6 Month Balance Sheet Net	Fixed
BSN12	12 Month Balance Sheet Net	Fixed
RATIO.P	Slaughter Ratio – Pig Crop (section 4.4)	Estimated
RATIO.M	Slaughter Ratio – Market Hogs (section 0)	Estimated
SL5WKS	5 Week Slaughter Ratio (section 4.6)	Estimated
ADMW.SL	Multiframe Survey result for litter rate	Fixed
ADXX.SL	List frame survey result for litter rate	Fixed
.5	List frame survey result for S_t/B_{t-1}	Fixed
1	Breeding Herd Smoother (section 10.3.2)	Fixed

13.3 Treatment 3 – ASB Measurements as Unbiased Estimates

For Treatment 3, we remove the ASB bias component from the system equations. The filter treats the ASB measurements as unbiased estimates of true inventory. As the ASB estimates are parameterized as unbiased and they give heavy weight to the survey litter rate, there is no longer a need to fix the variances associated with the litter rates noise processes and they are therefore estimated. Lastly, the ratio observation of survey sows farrowed to breeding herd is replaced by the ASB values and the variances is estimated instead of fixed. These changes are bolded in Table 9. All other measurements remain the same.

Table 9

Notation	Description	Parameterization Of Observation Noise
HP	ASB estimate for total hogs and pigs	Estimated
PP	ASB estimate for pig crop	Estimated
SP	ASB estimate for sows farrowed	Estimated
F1.P	ASB estimate for f_1	Estimated
F2.P	ASB estimate for f_2	Estimated
F3.P	ASB estimate for f_3	Estimated
F4.P	ASB estimate for f_4	Estimated
ADXX.H	List frame survey result for total hogs and pigs	Estimated
ADXX.P	List frame survey result for pig crop	Estimated
ADXX.S	List frame survey result for sows farrowed	Estimated
ADXX.F1	List frame survey result for f_1	Estimated
ADXX.F2	List frame survey result for f_2	Estimated
ADXX.F3	List frame survey result for f_3	Estimated
ADXX.F4	List frame survey result for f_4	Estimated
BSN3	3 Month Balance Sheet Net	Fixed
BSN6	6 Month Balance Sheet Net	Fixed
BSN12	12 Month Balance Sheet Net	Fixed
RATIO.P	Slaughter Ratio – Pig Crop (section 4.4)	Estimated
RATIO.M	Slaughter Ratio – Market Hogs (section 0)	Estimated
SL5WKS	5 Week Slaughter Ratio (section 4.6)	Estimated
ADMW.SL	**Multiframe Survey result for litter rate**	**Estimated**
ADXX.SL	**List frame survey result for litter rate**	**Estimated**
.5	**ASB estimate for S_t/B_{t-1}**	**Estimated**
1	Breeding Herd Smoother (section 10.3.2)	Fixed

13.4 Decomposition of $E[x_n|y_n]$ into Absolute Relative Net Contributions

In addition to comparing the fixed interval Kalman Smoother estimates of the *state* and their associated variances, we also examine the decomposition of the estimates. This is a way of comparing between the treatments which observations are most influential. For the sake of simplicity and relevance, we will look at the decomposition of the measurement contributions for the most recent measurement vector in time ($t = n$) for all three treatments and inventory items which are linear functions of the *state*. Specifically these are total hogs and pigs, pig crop, sows farrowed, and the weight group functions (the weight groups themselves are nonlinear functions of the *state*). As a measurement of inventory can have a positive or negative contribution, we will compare the absolute relative net contributions. We categorize the measurements according to measurement type. The six categories are ASB measurements, SURVEY measurements, BALANCE sheet measurements, SLAUGHTER ratio measurements, the cumulative historical contribution which will be explained in this section and which we will call “MODEL”, and OTHER measurements.

Table **10** shows the breakdown of observations into categories.

Table 10

Observation	Category
HP	ASB
PP	
SP	
F1.P	
F2.P	
F3.P	
F4.P	
ADXX.H	SURVEY
ADXX.P	
ADXX.S	
ADXX.F1	
ADXX.F2	
ADXX.F3	
ADXX.F4	
ADXX.SL	
ADMW.SL	
BSN3	BALANCE
BSN6	
BSN12	
RATIO.P	SLAUGHTER
RATIO.M	
SL5WKS	
.5	OTHER
1	
	MODEL

The “other” category consists of the sows farrowed/breeding herd ratio and the breeding herd constraint. We define a matrix M where the columns are the six categories and the rows are the observations. The values of M are 1 if the observation belongs to the category and zero otherwise. Define a matrix C where the columns are the seven inventory items H, P, S, F_1, F_2, F_3, and F_4; and the rows are the observations. The values of C contain the contribution of the observations such that the column sums equal the fixed interval Kalman Smoother estimates. Let **1** be a vector of ones with dimensions 6×1 . The absolute relative net contribution by category (ARNC) is then calculated as

$$ARNC = \frac{|M'C|}{\mathbf{11}'|M'C|} \tag{48}$$

The $ARNC$ matrix contains the absolute relative net contributions by inventory item (columns) and category (rows). Note that the brackets in this case stand for the absolute value and not the matrix determinate.

We now derive the matrix C containing the net contribution by inventory item (columns) and observation (rows). For a linear system of transition and observation equations, the Kalman Filter and Kalman smoother at $t = n$ provided in Shumway & Stoffer (2006) is

$$x_{n|n} = x_{n|n-1} + K_n\left(y_n - A_n x_{n|n-1}\right) \tag{49}$$

where $K_n = P_{n|n-1}A'_n\left(A_nP_{n|n-1}A'_n + R\right)^{-1}$ is the Kalman Gain and
$P_{n|n-1} = E\left[\left(x_n - x_{n|n-1}\right)\left(x_n - x_{n|n-1}\right)'|y_{n-1}\right]$ is the variance of the *state* prediction $x_{n|n-1}$. We can rewrite equation (49) as

$$x_{n|n} = (I - K_nA_n)x_{n|n-1} + K_ny_n \tag{50}$$

From the State-Space transition equation $x_{n|n-1} = \Phi x_{n-1|n-1}$, we can write equation (50) which is the Kalman Filter and Smoother at $t = n$ as the recursive relationship

$$x_{n|n} = (I - K_nA_n)\Phi x_{n-1|n-1} + K_ny_n \tag{51}$$

The contribution $(I - K_nA_n)\Phi$ in the first term of equation (51) weights the previous quarter's state $x_{n-1|n-1}$, and the contribution K_n from the second term weights the most recent measurements of inventory y_n. As this is a recursive equation, we will show that we can write this equation in terms of all measurements and the initial *state* $x_{0|0}$. We start with the recursive equation (51) and calculate $x_{n-1|n-1}$ as

$$x_{n-1|n-1} = (I - K_{n-1}A_{n-1})\Phi x_{n-2|n-2} + K_{n-1}y_{n-1} \tag{52}$$

Substitution of (52) into (51) yields

$$x_{n|n} = (I - K_nA_n)\Phi(I - K_{n-1}A_{n-1})\Phi x_{n-2|n-2} + (I - K_nA_n)\Phi K_{n-1}y_{n-1} + K_ny_n \tag{53}$$

With an additional third recursion, the pattern is recognizable

$$\begin{aligned} x_{n|n} = {} & (I - K_nA_n)\Phi(I - K_{n-1}A_{n-1})\Phi(I - K_{n-2}A_{n-2})\Phi x_{n-3|n-3} \\ & + (I - K_nA_n)\Phi(I - K_{n-1}A_{n-1})\Phi K_{n-2}y_{n-2} + (I - K_nA_n)\Phi K_{n-1}y_{n-1} + K_ny_n \end{aligned} \tag{54}$$

and it becomes evident that $x_{n|n}$ can be written

$$x_{n|n} = \left[\prod_{k=0}^{n-1}(I - K_{n-k}A_{n-k})\Phi\right]x_{0|0} + \sum_{m=0}^{n-1}\left[\prod_{j=1}^{m}\left(I - K_{n-j+1}A_{n-j+1}\right)\Phi\right]K_{n-m}y_{n-m} \tag{55}$$

For the initial *state* $x_{0|0}$ in equation (55) we are using published data with absolute certainty (initial variance of zero). We will call this $y_0 = x_{0|0}$. The result is that we can write the final Kalman Filter/Fixed interval Smoother estimate at $t = n$ as a "weighted average" of the initial *state* and all measurements.

$$x_{n|n} = \sum_{m=0}^{n}\lambda_m y_{n-m} \tag{56}$$

$$\lambda_m = \begin{cases} K_{n-m} & m = 0 \\ \left[\prod_{j=1}^{m}\left(I - K_{n-j+1}A_{n-j+1}\right)\Phi\right]K_{n-m} & 0 < m < n \\ \prod_{k=0}^{n-1}(I - K_{n-k}A_{n-k})\Phi & m = n \end{cases} \tag{57}$$

The interpretation is that the estimate $x_{n|n}$ is a composite of the initial *state* and all measurements in time weighted by the state-space model parameters. For an analysis of the decomposition, we will partition out from the summation in equation (56) the most recent data vector at $m = 0$

$$x_{n|n} = K_n y_n + \sum_{m=1}^{n} \lambda_m y_{n-m} \tag{58}$$

The first term of equation (58) is the contribution of the most recent measurement vector y_n; the second term is the composite contribution from the initial *state* and historical measurement vectors. This aggregate contribution of the historical observations is the "Model" category for the absolute relative net contributions. Each term's contribution in the summation could be calculated using the appropriate λ_m. For the nonlinear version of the decomposition, we have the Filter equations

$$\begin{aligned} x_{n|n} &= x_{n|n-1} + K_n\left[y_n - H\left(x_{n|n-1}\right)\right] \\ x_{n|n-1} &= G\left(x_{n-1|n-1}\right) \end{aligned} \tag{59}$$

The result is the recursion equation

$$\begin{aligned} x_{n|n} &= G\left(x_{n-1|n-1}\right) + K_n\left\{y_n - H\left[G\left(x_{n-1|n-1}\right)\right]\right\} \\ &= G\left(x_{n-1|n-1}\right) - K_n H\left[G\left(x_{n-1|n-1}\right)\right] + K_n y_n \\ &= J\left(x_{n-1|n-1}\right) + K_n y_n \end{aligned} \tag{60}$$

where $K_n y_n$ contains the contribution of the most recent data and $J\left(x_{n-1|n-1}\right)$ contains the contributions of cumulative past measurements analogous with the linear weighted average component $\sum_{m=1}^{n} \lambda_m y_{n-m}$. The contribution matrix is calculated as

$$C = \left[(K_n \times 1y_n')||(I - K_n A^*)x_{n|n-1}\right]' A^{*\prime} \tag{61}$$

where $\times$ is the elementwise multiplication operator, $||$ appends matrix columns, 1 is an $m \times 1$ vector of ones in which m is the dimension of the *state*, and A^* represents the linear transformation for which $A^* x_{n|n} = \left[H_n\ P_n\ S_n\ f_{1_n}\ f_{2_n}\ f_{3_n}\ f_{4_n}\right]'$. Substitution of (**61**) into (**48**) gives the Absolute Relative Net Contribution matrix.

$$ARNC = \frac{\left|M'\left[(K_n \times 1y_n')||(I - K_n A^*)x_{n|n-1}\right]' A^{*\prime}\right|}{\mathbf{11}'\left|M'\left[(K_n \times 1y_n')||(I - K_n A^*)x_{n|n-1}\right]' A^{*\prime}\right|} \tag{62}$$

13.5 Contribution Tables

The remainder of this paper contains the results of hog inventory estimation through Signal Filtering based on the three treatments defined in sections 13.1 - 13.3. Table 11 contains the relative absolute net contribution tables for each of the three treatments by categorized measurements; ASB, survey, balance sheet data, slaughter data, other (composed of sows to breeding herd ratio and breeding herd smoother), and historical observations (called "Model") . Following these tables are graphs of the estimates and charts of the contribution allocations organized by inventory item, as well as graphs of the constraints. It should be noted that because the natural log of 1 is zero, the contribution of the breeding herd smoother is incalculable.

Table 11

Table 12: Summary Contribution Equally Weighted over all Inventory Intentory Items

Figure 36: Total Hogs and Pigs

Figure 37: Pig Crop

Figure 38: Sows Farrowed

Figure 39**:** Weight Group Function 1

Figure 40**:** Weight Group Function 2

Figure 41: Weight Group Function 3

Figure 42: Weight Group Function 4

Figure 47: Total Market Hogs

Figure 49: Litter Rate

Figure 51: Weight Group Ratio Transition

Figure 53**:** Market Hogs and Slaughter Ratio

Figure 55**:** Three Month Balance Sheet Residual

Figure 57**:** Twelve Month Balance Sheet Residual

13.6 Summary of Results

Treatments 1, 2, and 3 demonstrate that it is possible to start at the same level of inventory at a particular point in time and generate many different solutions, given the same constraints that the ASB uses to set its published numbers. The variance parameters Q and R were estimated to maximize the likelihood of the realization of the observation vectors, given the constraints and assumptions discussed in this paper. If a particular constraint - for example, a slaughter ratio - is not deemed adequate, the corresponding noise process in these covariance matrices could be fixed, as was done with the survey litter rates in order to enforce ASB behavior in the treatments in which there were no ASB measurements, or in which the ASB measurements were parameterized to contain possible bias. Strictly enforcing too many constraints can have computational complications, as a solution may not in fact exist that satisfies all constraints.

13.6.1 Treatment 1 – No Expert Measurements

The plots of the fixed interval Kalman Smoother estimates in Figure 36 through Figure 49 show that the results for pig crop, sows farrowed, weight groups 1 and 2, and the litter rate appear to agree for the most part with the ASB published inventories. Total hog inventory, total market hogs, and consequently breeding herd appear significantly different. Further inspection shows that these differences are attributed to weight groups 3 and 4 (Figure 45 and Figure 46). These weight groups are encapsulated in weight group function 3 (Figure 41), which also deviates considerably from the ASB published. Both the survey results and ASB measurements for f_3 and f_4 will prove to be highly influential observations in treatments 2 and 3 relative to the other measurements.

One notable result is the slaughter ratios' lack of contribution to the estimates in all three treatments. We are examining the influence of the measurements of inventory at the last point in time on the estimates of inventory at that time. In section 4.4 it was established that slaughter has a lagged effect on inventory by two quarters. One would expect that the slaughter ratios at time $t = n$ would have some influence on the inventory estimates at some point $n - 2 \leq t \leq n$. However this is not the case. The maximum relative absolute net contribution of the slaughter ratios in treatment 1 is 0.38% in the Kalman Smoother for total hogs and pigs at time $t = n$. In treatment 2 the maximum relative absolute net contribution of the slaughter ratios is 0.05% for f_4 at time $t = n$. In treatment 3 the maximum relative absolute net contribution of the slaughter ratios is 0.25% for the total hogs and pigs estimate at $t = n$.

13.6.2 Treatment 2 – Biased Expert Measurements

The initial *state* of the Filter at $x_{0|0} = E[x_0|y_0]$ for all three treatments is set at the published ASB values and is parameterized with zero uncertainty i.e. $var(x_{0|0}) = P_{0|0} = 0$. This means that for the initial starting position of each filter scenario, the ASB values are treated as absolute truth. The initial values of the bias parameters corresponding to each ASB measurement in treatment 2 are initialized at zero. The visually apparent biases in weight group 3 and weight group 4 seen in treatment 1 are also observable in the filter results of treatment 2. As treatment 2 estimates the ASB bias as part of the *state* and we estimate the variance of the *state* within the filter, we can test the null hypothesis that the bias is equal to zero versus the alternative that there exists a nonzero bias in the ASB estimates. Graphs of the p-values of this hypothesis test for each point in time are given in Figure 58 through Figure 64. P-values that fall below the dashed red line in the graphs indicate there is enough evidence to reject the null hypothesis that the bias is zero at that particular point in time at a significance level of $\alpha = 0.05$. This analysis does not take into account simultaneous inference across time and inventory items. Independent hypothesis testing of the Treatment 2 bias parameters for pig crop, Figure 59, and sows farrowed, Figure 60, shows greatest lack of evidence of bias for those two inventory items. There is sufficient evidence to support the existence of bias in the ASB measurements as time progresses for total hogs and pigs and the four weight group functions.

Figure 58

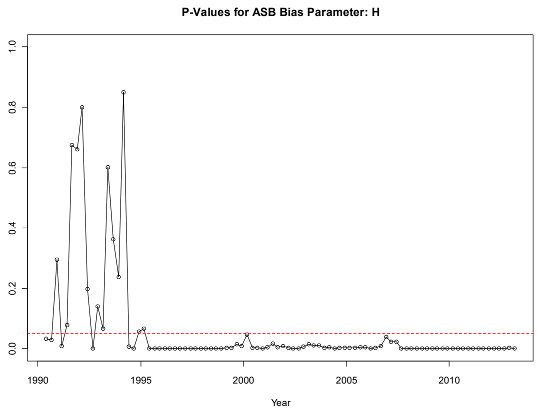


Figure 59

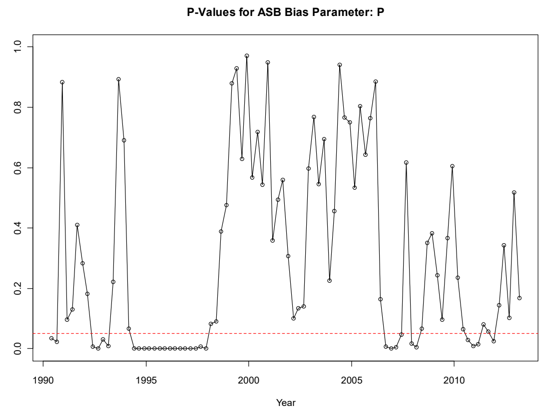


Figure 60

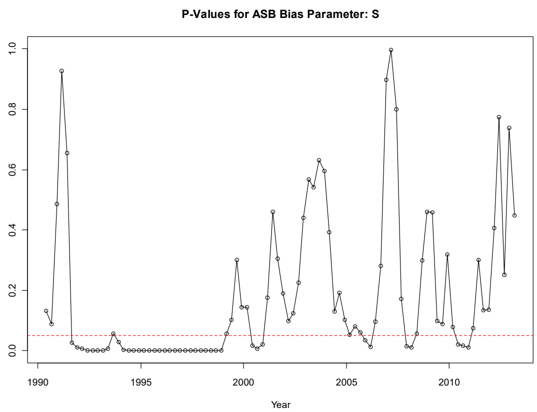


Figure 61

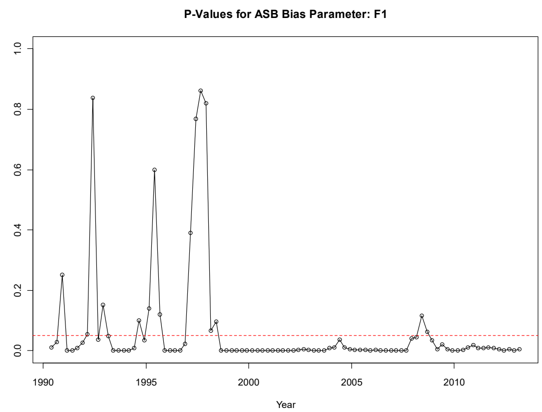


Figure 62

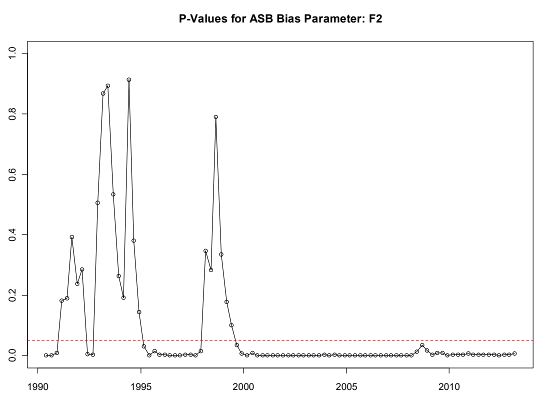


Figure 63

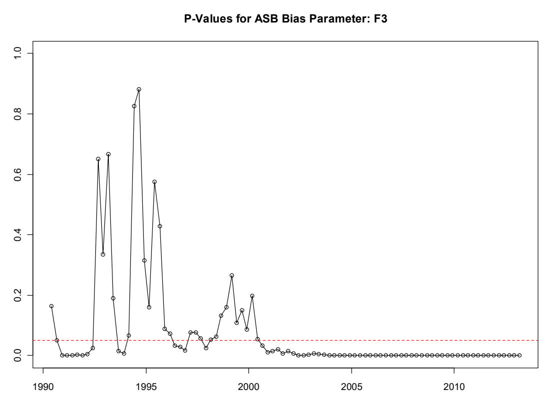
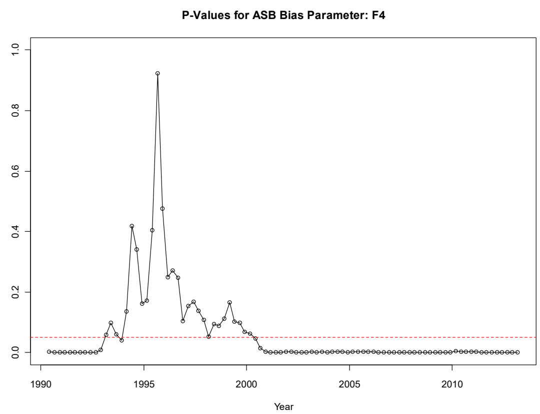


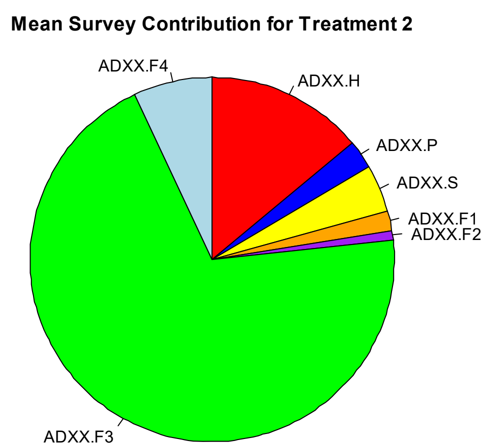
Figure 64



The mean relative absolute net contribution by category for Treatment 2 in Table **12** on page 40 shows that both the survey results and the ASB measurements are highly influential measurements in the inventory estimates. The survey results contribute on average 42.28% of the estimates and the ASB measurements contribute 44.84% of the estimates. Of these percentages, the survey results and ASB estimates for weight group functions 3 and 4 make up over 75% of the survey and ASB relative absolute net contributions to the inventory item estimates on average (see Figure 65 and Figure 66). These functions contain weight groups 3 and 4 which by visual inspection contain the most relative bias. Intuitively it could be hypothesized that weight group 3 is not restricted by as many constraints as the other inventory items, and therefore the filter relies heavily on observations containing information on weight group 3. It is also worth noting the presence of bias in the ASB measurement for weight group 4 and weight group function 4 (Figure 46, middle) despite the "good behavior" of the five week slaughter ratio relative to the annual ratio of weight group 4 (Figure 54, middle).

Figure 65

Figure 66



13.6.3 Treatment 3 – Unbiased Expert Measurements

The results for the final parameterization of ASB measurements as unbiased estimates of true inventory demonstrate that congruency can be achieved with external data and the inventory estimates, in addition to a smooth transition from the publishing of ASB measurements to Kalman Smoother measurements. Treatment 3 in Table 12 on page 40 shows the categorized mean relative absolute net contribution of each measurement type to hog inventories. In comparing Treatment 3 to Treatment 2 with regard to the measurement contributions, it appears that treating the ASB measurements as unbiased resulted in a more uniform overall contribution of measurements on the final inventory estimates than in the case of the biased ASB measurements (Treatment 2, Table 12). The absolute relative contribution of the historical estimates (labeled "model") on the last inventory measurements increased from the Treatment 2 level. In Figure 67 and Figure 68 we see that the most influential survey results and ASB measurements are weight group functions 3 and 4, similar to the contribution profile of Treatment 2. When the ASB measurements are treated as unbiased, the ASB measurement for weight group function 4 shows comparable influence to that of weight group function 3 (Figure 68).

Figure 67

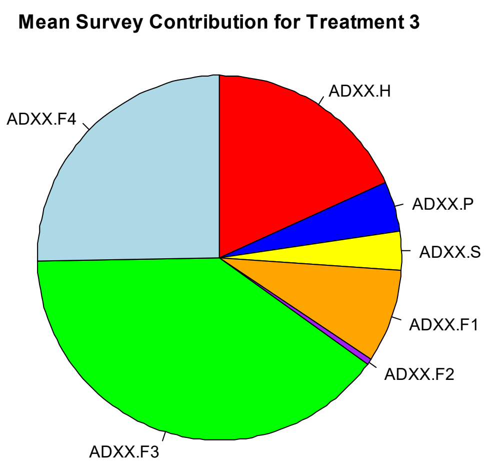
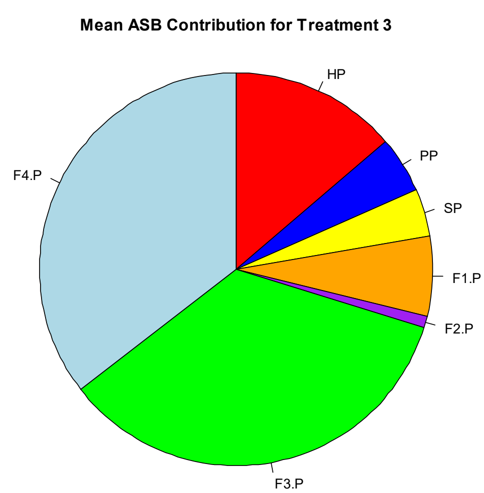
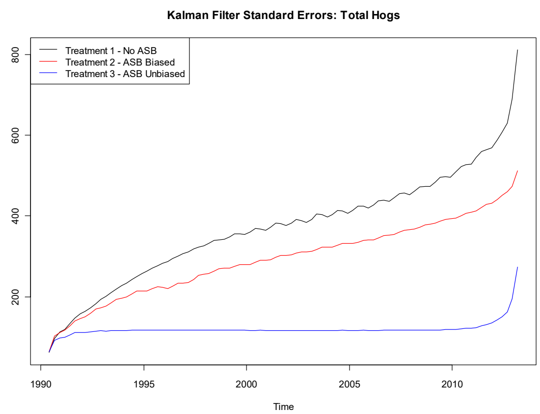


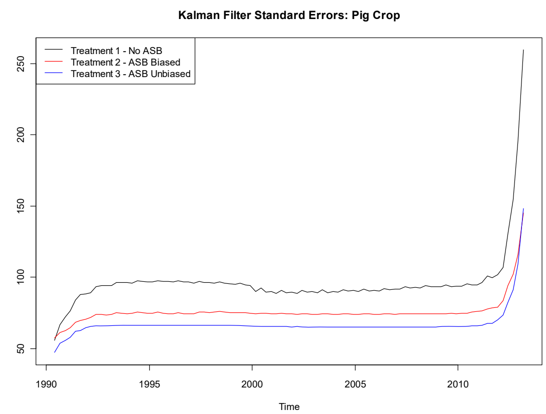
Figure 68

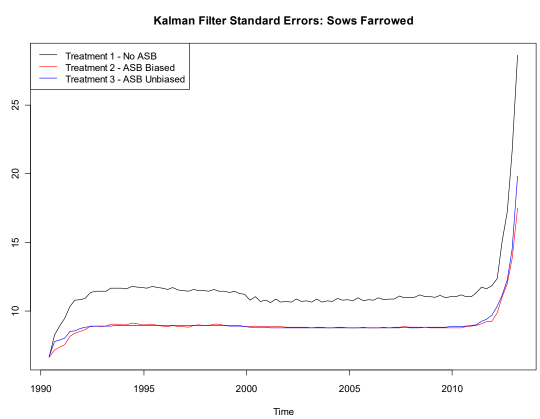


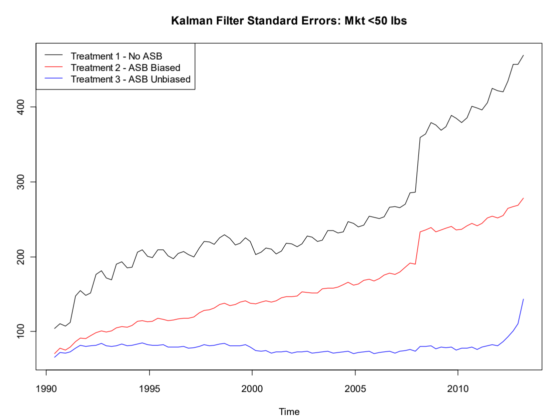
13.7 Standard Errors

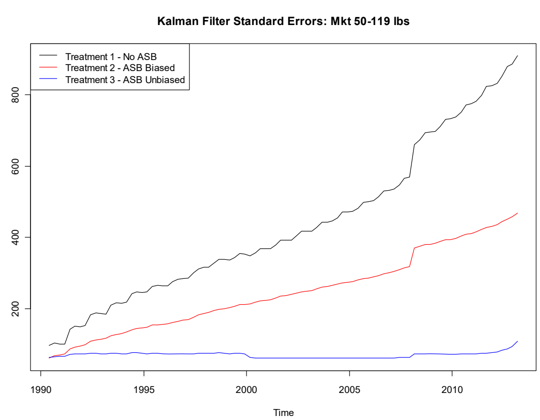
Graphs of the standard errors that follow demonstrate that when the ASB measurement is included and treated as an unbiased estimate, the fixed interval Kalman Smoother variance estimates are minimized between all three treatments. For Sows Farrowed, the standard errors are very close between Treatments 2 and 3. For all other inventory items, Treatment 3 resulted in the smallest standard errors. Excluding expert opinion resulted in the highest standard errors of the fixed interval Kalman Smoother estimates.

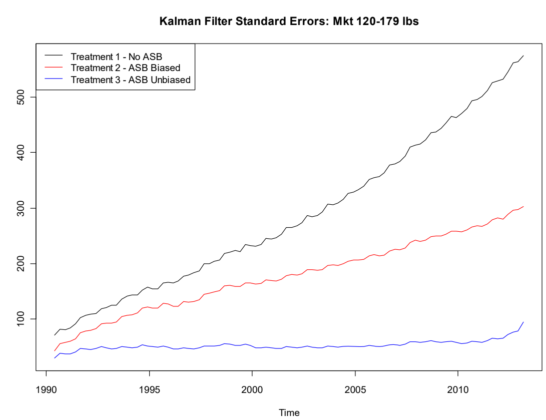

Kalman Filter Standard Errors: Total Hogs
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
200
400
600
800
1990
1995
2000
2005
2010
Time

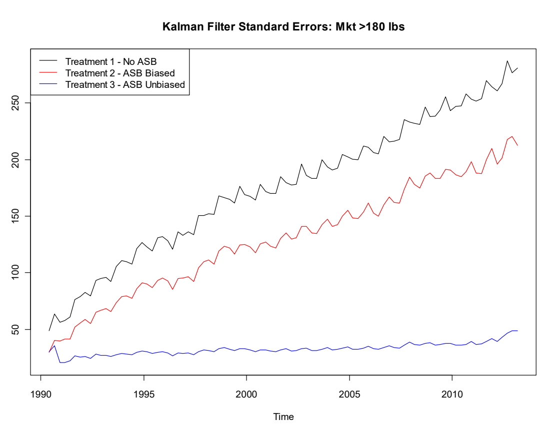

Kalman Filter Standard Errors: Pig Crop
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
50
100
150
200
250
1990
1995
2000
2005
2010
Time

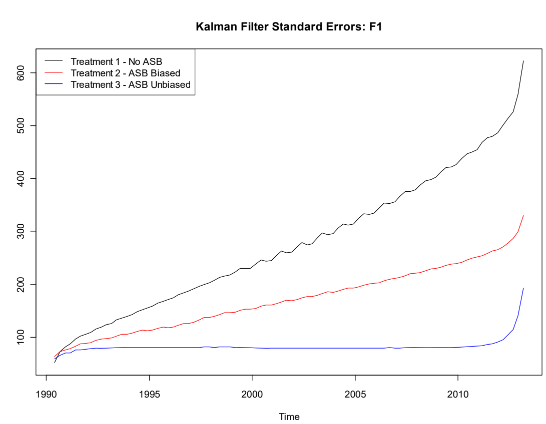

Kalman Filter Standard Errors: Sows Farrowed
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
10
15
20
25
1990
1995
2000
2005
2010
Time

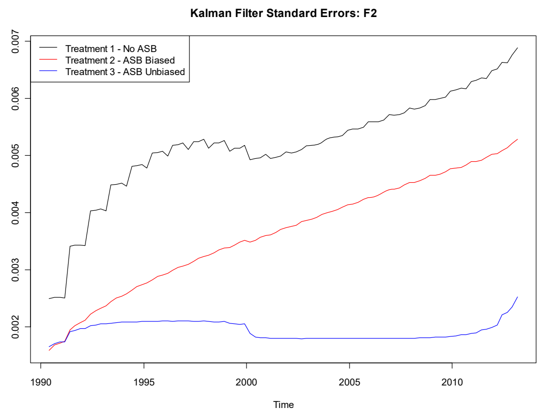

Kalman Filter Standard Errors: Mkt <50 lbs
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
100
200
300
400
1990
1995
2000
2005
2010
Time

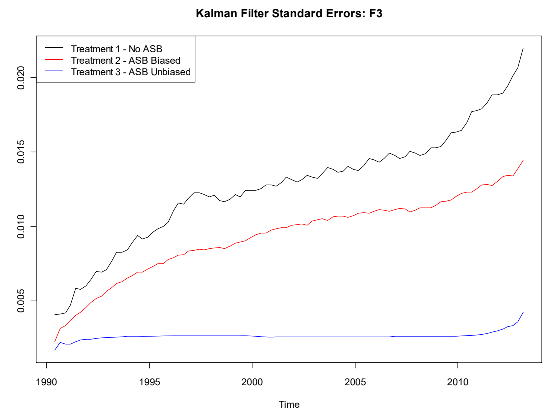

Kalman Filter Standard Errors: Mkt 50-119 lbs
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
200
400
600
800
1990
1995
2000
2005
2010
Time

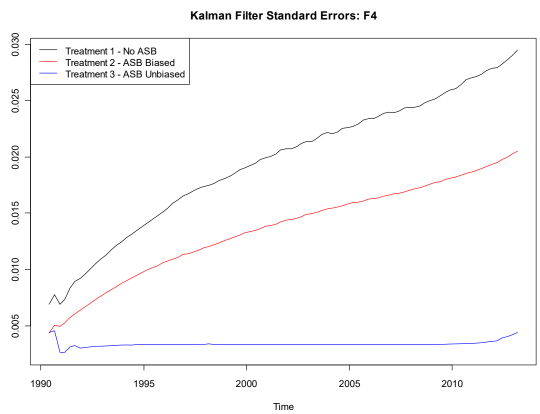

Kalman Filter Standard Errors: Mkt 120-179 lbs
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
100
200
300
400
500
1990
1995
2000
2005
2010
Time


Kalman Filter Standard Errors: Mkt >180 lbs
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
50
100
150
200
250
1990
1995
2000
2005
2010
Time


Kalman Filter Standard Errors: F1
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
100
200
300
400
500
600
1990
1995
2000
2005
2010
Time


Kalman Filter Standard Errors: F2
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
Time


Kalman Filter Standard Errors: F3
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
Time


Kalman Filter Standard Errors: F4
Treatment 1 - No ASB
Treatment 2 - ASB Biased
Treatment 3 - ASB Unbiased
Time

References

Anderson, D.O. and Moore, John B. *Optimal Filtering.* Englewood Cliffs, NJ: Prentice-Hall Inc., 1979.

Casella, George and Berger, Roger L. Statistical Inference. Pacific Grove, CA: the Wadsworth Group, 2002.

Durbin, J. and Koopman, S.J. *Time Series Analysis by State Space Methods.* Oxford: Oxford University Press, 2012.

Green, W. H. *Econometric Analysis (4th Edition)*. New Jersey: Prentice-Hall, 2000.

Shumway, R.H. and Stoffer, D.S. *Time Series Analysis and its Applications*. New York: Springer Science+Business Media, LLC., 2006.