Hierarchical models in the production of official statistics: a discussion of some practical aspects

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Why hierarchical models in the production of official statistics?

- need to account for various sources of error
  - sampling error, measurement error, linking error
- need for a transparent, reproducible and validated process
  - analytic techniques
- need for measures of uncertainty
  - entire distribution
Hierarchical model - general form

Data model: $[y|\theta, \psi]$

Process model: $[\theta|\psi]$

Parameter model: $[\psi]$

Joint distribution

$$[y, \theta, \psi] = [y|\theta, \psi][\theta|\psi][\psi]$$

Predictive distribution

$$[\theta|y, \psi]$$

Three cases

- known $\psi$
- unknown, fixed $\psi$
- unknown, random $\psi$
Why model-based small area estimation (SAE) in the production of official statistics?

- need to account for various sources of error
- need for a transparent, reproducible and validated process
- need for measures of uncertainty
- integration of data from multiple sources
  - observed survey data within area + auxiliary data within area + information across all areas
- growing demand for granular statistics
  - quantities of interest: i.e. totals, means, ratios
  - ‘area’ / ‘domain’: i.e. geography, socio-economic status, occupation
- increasing costs of data collection
  - ‘small’: amount of survey data available for estimation within a given area (realized sample size as small as zero)
Small domain hierarchical model - general form

**Sampling model:** \([y_i | \theta_i, x_i, \psi_{\theta,i}, \psi_{y,i}]\)

**Linking model:** \([\theta_i | x_i, \psi_{\theta}]\)

**Parameter model:** \([\psi_{\theta,i}, \psi_{y,i}]\)

**Joint distribution**

\[ [y_i, \theta_i, x_i, \psi_{\theta}, \psi_{y,i}] = [y_i | \theta_i, x_i, \psi_{\theta}, \psi_{y,i}][\theta_i | x_i, \psi_{\theta}][\psi_{\theta}, \psi_{y,i}] \]

**Predictive distribution**

\[ [\theta_i | y_i, x_i, \psi_{\theta}, \psi_{y,i}] \]

- *typically* known \(x_i, \psi_{y,i}\) (area-level models)
- still three cases
  - known \(\psi_{\theta}\)
  - unknown, fixed \(\psi_{\theta}\)
  - unknown, random \(\psi_{\theta}\)
Examples of model-based SAE in government programs

- **Chilean Ministerio de Desarrollo, World Bank**: poverty mapping
  - Casas Cordero Valencia et al. (2016); Elbers et al. (2003)
- **Census Bureau**: income and poverty measures
  - Bell et al. (2016)
- **National Agricultural Statistics Service**:
  - cash rental rates, Erciulescu et al. (2018)
  - crops production, Erciulescu et al. (2019a)
  - agricultural labor wages, Erciulescu (2018)
- **Organisation for Economic Co-operation and Development**: adult competency
  - Krenzke et al. (2019)
- **Bureau of Labor Statistics**: employee compensation components
  - Erciulescu and Opsomer (2019)
A small domain hierarchical Bayes multivariate model for employee compensation components*

Sampling Model: \[ y_i | \theta_i \sim N(\theta_i, \Sigma_{ei}) \]

Linking Model: \[ \theta_i | (\beta, \Sigma_v) \sim N(x_i \beta, \Sigma_v) \]

Independent priors: \[ \pi(\beta, \Sigma_v) = \pi(\beta)\pi(\Sigma_v) \]

- domain \( i \), cross-tabulation of census divisions, 6-digit SOC system codes, work levels, binary characteristics
- \( \theta_i \), quantities of interest, wage and benefits
- \( y_i \), direct survey estimates, domain-level wage and benefits direct survey estimates
- \( x_i \), known covariates, selected using sample domains definitions
- \( \beta \), regression coefficients
- \( \Sigma_v \), linking model variance-covariance matrix
- \( \Sigma_{ei} \), known survey variance-covariance matrices

* Erciulescu and Opsomer, 2019
A common approach for fit and prediction

Fit
- Markov chain Monte Carlo (MCMC): multiple chains, iterations, burn-in, thining; keep R samples for inference

Prediction
- in-sample domain $i$
  - $R$ samples $\theta_{i\zeta}, \zeta = 1, \cdots, R$, from $[\theta_i | (y_i, \psi_\theta, \psi_y)]$
- not-in-sample domain $i'$:
  - generate new $R$ samples $\theta_{i'\zeta}, \zeta = 1, \cdots, R$, from $[\theta_i | x_i, \psi_\theta, \zeta]$
- small domain posterior means: $R^{-1} \sum_{\zeta=1}^{R} \theta_{i\zeta}$
- small domain posterior variances: $R^{-1} \sum_{\zeta=1}^{R} \left( \theta_{i\zeta} - R^{-1} \sum_{\zeta=1}^{R} \theta_{i\zeta} \right)^2$
- small domain posterior $p$ quantiles: $\theta_{i(p)}, \theta_i = (\theta_{i1}, \cdots, \theta_{iR})$

Practical challenges: number of MCMC samples, large number of domains (in-sample and not-in-sample), variable selection
Prediction for other functions

For example,

- sum of $\theta_i$ components (total employee compensation, proportions of adult literacy/numeracy scores in prespecified ranges)
  - inference using $R$ samples $\theta^S_{i\zeta} := \theta_{i\zeta,1} + \theta_{i\zeta,2}, \zeta = 1, \cdots, R$

- ratio of $\theta_i$ components (labor wage as ratio of total income to total hours worked, crop yield as the ratio of total production to total harvested acres)
  - inference using $R$ samples $\theta^R_{i\zeta} := \theta_{i\zeta,1}/\theta_{i\zeta,2}, \zeta = 1, \cdots, R$

- aggregates of $\theta_i$ for pre-defined domains (2-digits SOC codes, county-agricultural district-state-census division-nation)
  - inference using $R$ samples $\theta^D_{i\zeta} := \sum_{i \in \text{pre-defined domain}} \theta_{i\zeta}, \zeta = 1, \cdots, R$
Examples of model validation

Internal

- mixing and convergence diagnostics: \( \hat{R} \), MC effective sample size, autocorrelation
- residuals diagnostics: unconditional and conditional
- posterior predictive checks: indicator, correlation, deviance, residuals
- alternative model specifications: prior distributions

External

- predictions versus direct estimates
- predictions for not-in-sample domains versus predictions for in-sample domains
- cross-validation

Practical challenges: autocorrelation, cross-validation, visualization, storage
Simulation study: Data generation model

\[ y_i | \theta_i \sim N(\theta_i, \Sigma_{ei}) \]

\[ \theta_i | (\beta, \Sigma_v) \sim N(x_i \beta, \Sigma_v) \]

\[ \pi(\beta, \Sigma_v) = \pi(\beta) \pi(\Sigma_v) \]

- \( i = 1, \ldots, m \)
- \( y_i = (y_{i,1}, y_{i,2}) \)
- \( x_i, \) two identical rows \( x_{i,\text{row}} = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,(p+1)}) \)
  - \( x_{i,1} = 1, x_{i,2} \sim \text{Beta}(2, 4), x_{i,k} \sim N(\mu_{xk}, \sigma_{xk}^2), k = 3, \ldots, (p + 1) \)
  - \( \mu_{xk} \sim \text{Unif}(1, 50), \sigma_{xk} \sim \text{Unif}(1, 10), k = 3, \ldots, (p + 1) \)
- \( \beta = (\beta_1, \beta_2), \beta_j = (\beta_{j,1}, \beta_{j,2}, \beta_{j,3}, \ldots, \beta_{j,(p+1)}), j = 1, 2 \)
  - \( \beta_{1,1} = 1, \beta_{1,2} = 10, \beta_{1,k} \sim \text{Unif}(1, 5) + N(0, 1), k = 3, \ldots, (p + 1) \)
  - \( \beta_{2,1} = 1, \beta_{2,2} = 10, \beta_{2,k} \sim \text{Unif}(1, 2) + N(0, 1), k = 3, \ldots, (p + 1) \)
- \( \Sigma_v \sim \text{inverse - Wishart}(3, I_2) \)
- \( \text{diag}(\Sigma_{ei}) = (\sigma_{ei,1}^2, \sigma_{ei,2}^2), \sigma_{ei,j}^2 = \exp(\log(x_{i,j} \beta_j) + N(0, 1)), j = 1, 2 \)
- \( \text{cor}(y_{i,1}, y_{i,2}) \sim \text{Unif}(0.3, 0.9) \)
Simulation study: Fitted models

\[ y_i | \theta_i \sim N(\theta_i, \Sigma_e) \]

\[ \theta_i | (\beta, \Sigma_v) \sim N(x_i \beta, \Sigma_v) \iff \theta_i | (\beta, v_i) = x_i \beta + v_i, v_i | \Sigma_v \sim N(0, \Sigma_v) \]

\[ \pi(\beta, \Sigma_v) = \pi(\beta) \pi(\Sigma_v) \]

Software

- R STAN, R JAGS

Bayesian specification

- hierarchical Bayes: \( \pi(\beta) = N(0, 10^4) \) and \( \Sigma_v \sim \text{Inverse} - \text{Wishart}(3, I) \)

- empirical Bayes: \( \pi(\beta_1) = N(0, 10^4) \); \( \beta_{-1} \) set equal to the least squares estimates, based on a multiple linear regression model and \( \Sigma_v \sim \text{Inverse} - \text{Wishart}(3, I) \)

Practical challenges: prior distributions for the linking model variance-covariance components (Inverse-Gamma/Uniform/Cauchy/F; Inverse-Wishart/LKJ)
Simulation study: Fitted models specifications

<table>
<thead>
<tr>
<th>Parameters $(m, p)$</th>
<th>Model</th>
<th>Software</th>
<th># MC samples / chain (start, burn-in, thin)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(100, 10)$</td>
<td>S1</td>
<td>STAN</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J1</td>
<td>JAGS</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J1l</td>
<td>JAGS</td>
<td>$(30000, 10000, 10)$</td>
</tr>
<tr>
<td>$(1000, 100)$</td>
<td>S2</td>
<td>STAN</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J2</td>
<td>JAGS</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J2l</td>
<td>JAGS</td>
<td>$(30000, 10000, 10)$</td>
</tr>
<tr>
<td>$(10000, 100)$</td>
<td>S3</td>
<td>STAN</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J3</td>
<td>JAGS</td>
<td>$(3000, 1000, 10)$</td>
</tr>
<tr>
<td></td>
<td>J3l</td>
<td>JAGS</td>
<td>$(30000, 10000, 10)$</td>
</tr>
</tbody>
</table>
Simulation study: JAGS HB model specification

```r
model{
  for(i in 1:m){
    y[i,1:C] ~ dmmnorm(theta[i,1:C], Sigmae.inv[i,1:C,1:C])
    theta[i,1:C] = thetai0[i,1:C] + v[i,1:C]
    v[i,1:C] ~ dmmnorm(muv[1:C], Sigmav.inv[1:C,1:C])
  }
  for (c in 1:C) {
    theta0[i,c] = X[i,1:P]%*%beta[1:P,c]
  }
  Sigmae.inv[i,1:C,1:C] = inverse(Sigmae[i,1:C,1:C])
}

## Priors:
for (k in 1:p){
  for (c in 1:C){
    beta[k,c] ~ dnorm(0, 1/100)
  }
}
Sigmav.inv ~ dwish(Kv, 3)
Sigmav = inverse(Sigmav.inv)
```
Simulation study: JAGS EB model specification

model{
  for(i in 1:m){
    y[i,1:C] ~ dmnorm(theta[i,1:C], Sigmae.inv[i,1:C,1:C])
    theta[i,1:C] = theta0[i,1:C] + v[i,1:C]
    v[i,1:C] ~ dmnorm(muv[1:C], Sigmav.inv[1:C,1:C])
  }

  for (c in 1:C) {
    theta0[i,c] = x[i,1]*beta[c] + x[i,2:p]*betaF[2:p,c]
  }

  Sigmae.inv[i,1:C,1:C] = inverse(Sigmae[i,1:C,1:C])
}

for (c in 1:C){
  beta[c] ~ dnorm(0, 1/100)
}

Sigmav.inv ~ dwish(Kv, 3)
Sigmav = inverse(Sigmav.inv)
Simulation study: STAN HB model specification

data {
  int<lower=0> m;
  int<lower=0> p;
  real x[m,p];
  vector[2] y[m];
  cov_matrix[2] Sigmae[m];
}

parameters {
  real beta[p,2];
  vector[2] v[m];
  cov_matrix[2] Sigma;
}

transformed parameters {
  vector[2] theta[m];
  for (i in 1:m)
    vector[2] theta0;
    for (k in 1:2) 
      theta0[k] = 0;
    for (j in 1:p)
      theta0[k] = beta[j,k] = x[i,j];

    theta[i] = v[i] + theta0;
}

model {
  for (i in 1:p)
    for (k in 1:2)
      beta[i,k] ~ normal(0, 100);
  Sigmae = inv_wishart(3, K);
  for (i in 1:m)
    v[i] ~ multi_normal([0, 0], Sigmae);
  for (i in 1:m)
    y[i] ~ multi_normal(theta[i], Sigmae[i]);
}
Simulation study: STAN EB model specification

data{
  int<lower=0> m;
  int<lower=0> p;
  real x(m,p);
  vector[2] y[m];
  cov_matrix[2] Sigma[m];
  real beta[p,2];
}

parameters{
  real beta[2];
  vector[2] v[m];
  cov_matrix[2] sigmav;
}

transformed parameters{
  vector[2] theta[m];
  for (i in 1:m) {
    vector[2] theta0;
    for (k in 1:2) {
      theta0[k] = beta[k] * x[i,k];
      for (j in 2:p)
        theta0[k] += beta[j,k] * x[i,j];
    }
    theta[i] = v[i] + theta0;
  }
}

model{
  for (k in 1:2)
    beta[k] ~ normal(0, 100);
  sigmav ~ inv_wishart(3, K);
  for (i in 1:m)
    v[i] ~ multi_normal([0, 0], sigmav);
  for (i in 1:m)
    y[i] ~ multi_normal(theta[i], Sigma[i]);
}
Simulation study: Computational time results

Table 2: Computational Time Summaries

<table>
<thead>
<tr>
<th>Model</th>
<th>Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HB</td>
</tr>
<tr>
<td>S1</td>
<td>525</td>
</tr>
<tr>
<td>J1</td>
<td>5</td>
</tr>
<tr>
<td>J1l</td>
<td>35</td>
</tr>
<tr>
<td>S2</td>
<td>32744</td>
</tr>
<tr>
<td>J2</td>
<td>1438</td>
</tr>
<tr>
<td>J2l</td>
<td>14227</td>
</tr>
<tr>
<td>S3</td>
<td>354416</td>
</tr>
<tr>
<td>J3</td>
<td>18210</td>
</tr>
<tr>
<td>J3l</td>
<td>Too long ...</td>
</tr>
</tbody>
</table>
Simulation study: Convergence results

Table 3: Convergence Results Summaries HB

<table>
<thead>
<tr>
<th>Model</th>
<th>Approach</th>
<th>$\hat{R}$ (min, median, max)</th>
<th>MC Effective Sample Size (min, median, max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>HB</td>
<td>(0.995, 1.000, 1.072)</td>
<td>(44, 571, 825)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(0.995, 1.005, 1.086)</td>
<td>(35, 342, 810)</td>
</tr>
<tr>
<td>J1</td>
<td>HB</td>
<td>(0.999, 1.004, 1.070)</td>
<td>(44, 600, 600)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(0.999, 1.023, 1.216)</td>
<td>(14, 440, 600)</td>
</tr>
<tr>
<td>J1l</td>
<td>HB</td>
<td>(1.001, 1.001, 1.030)</td>
<td>(130, 5800, 6000)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(1.001, 1.003, 1.014)</td>
<td>(190, 3400, 7500)</td>
</tr>
<tr>
<td>S2</td>
<td>HB</td>
<td>(0.995, 0.999, 1.289)</td>
<td>(14, 580, 908)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(0.995, 0.999, 1.262)</td>
<td>(9, 583, 802)</td>
</tr>
<tr>
<td>J2</td>
<td>HB</td>
<td>(0.999, 1.010, 1.818)</td>
<td>(6, 600, 600)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(1.037, 1.116, 2.029)</td>
<td>(5, 79, 600)</td>
</tr>
<tr>
<td>J2l</td>
<td>HB</td>
<td>(1.001, 1.002, 1.140)</td>
<td>(33, 6000, 6000)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(1.001, 1.002, 1.138)</td>
<td>(59, 3500, 7500)</td>
</tr>
<tr>
<td>S3</td>
<td>HB</td>
<td>(0.995, 1.000, 4.097)</td>
<td>(2, 582, 989)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(0.995, 1.003, 2.427)</td>
<td>(3, 503, 1109)</td>
</tr>
<tr>
<td>J3</td>
<td>HB</td>
<td>(0.999, 1.008, 2.118)</td>
<td>(5, 600, 600)</td>
</tr>
<tr>
<td></td>
<td>EB</td>
<td>(0.999, 1.043, 3.003)</td>
<td>(1, 600, 600)</td>
</tr>
</tbody>
</table>
Example output from the small domain hierarchical Bayes multivariate model for employee compensation components*

- J1/2/3-type model ($m = 16,107; p = 18$)
- 16,107 survey estimates and in-sample model predictions, and 556,221 not-in-sample model predictions

* Erciulescu and Opsomer, 2019
Final thoughts...

- a flavor of innovation in official statistics programs
- existing tools and potential for development of novel ones
- software: more than just R JAGS and R STAN
- alternative sampling methods
- practical challenges: time + number of MCMC samples, large number of domains (in-sample and not-in-sample), variable selection, prior distributions for the linking model variance-covariance components, autocorrelation, cross-validation, visualization, storage
- beyond small area estimation - for example, bridging models for the Association of Fish and Wildlife Agencies (Erciulescu et al. 2019b)


Thank you!

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